On the Definition of Motives on Algebraic Varieties

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Abstract

Let $X$ be an algebraic variety over $\mathbb{C}$. Let $\bar{X} \to X$ be its resolution of singularity. Then the mixed motive of $X$ of degree $k$ ($k \in \mathbb{Z}$) is the set of harmonic $k$-forms $\omega$ on $\bar{X}$ such that $\int_\sigma \omega \in \mathbb{Q}$, where $\sigma$ is a singular $k$-cycle of $\bar{X}$.

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1 Introduction

In [4] Grothendieck proposed the concept of motives, ‘a decomposition of an algebraic variety over $\mathbb{C}$ into equidimensional pieces.’ Let $X$ be an algebraic variety over $\mathbb{C}$ and $\bar{X} \to X$ its resolution of singularity. With the use of mixed Hodge Conjecture (Theorem 18) the axioms of mixed Weil cohomology theory are formulated (see Definition 19). Then we finally define the mixed motive of degree $k$ ($k \in \mathbb{Z}$) as the set $\mathcal{M}_k(\bar{X}, \mathbb{Q})$ of harmonic $k$-forms $\omega$ on $\bar{X}$ such that $\int_\sigma \omega \in \mathbb{Q}$, where $\sigma$ is a singular $k$-cycle of $\bar{X}$.

2 Hodge Theory

Let $M$ be a compact manifold of real dimension $N$. Let $\mathcal{C}^K(M)$ ($0 \leq K \leq N$) be the set of $K$-forms on $M$. Let $*$ be the Hodge’s star operator. Define

$$\langle (\alpha, \beta) \rangle := \int_M \alpha \wedge *\beta \quad (\alpha, \beta \in \mathcal{C}^K(M)).$$

(1)

Let $d$ be the exterior derivative on $M$ and $\delta$ be its formal adjoint with respect to the above $\langle (\ , \ ) \rangle$. Let $\mathcal{H}^K(M, \mathbb{C})$ be the set of harmonic $K$-forms on $M$. The following is a special case of [3], Chapter VI, Section 3.3, (3.16) Theorem.
**Theorem 1.** Let $M$ be a compact manifold of dimension $N$. Then

$$\mathcal{E}^K(M) = \mathcal{H}^K(M, \mathbb{C}) \oplus \text{Im}d \oplus \text{Im}\delta \ (0 \leq K \leq N).$$  \hfill (2)

### 3 Mixed Hodge Conjecture

Let $X$ be a smooth algebraic variety over $\mathbb{C}$. By definition the set $\mathcal{H}^k(X, \mathbb{C})$ of harmonic $k$-forms on $X$ decomposes as follows.

$$\mathcal{H}^k(X, \mathbb{C}) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X, \mathbb{C}),$$  \hfill (3)

where $\mathcal{H}^{p,q}(X, \mathbb{C})$ is the set of harmonic $(p, q)$-forms on $X$. Let $0 \leq p \leq n := \text{dim}X$ be an integer. An algebraic $p$-piece is a finite formal $\mathbb{Q}$-linear combination $\sum_i c_i \Gamma_i (c_i \in \mathbb{Q})$ of irreducible $p$-dimensional algebraic subvarieties $\{\Gamma_i\}$ on affine open subsets of $X$. Let $C'_p(X)$ be the set of harmonic $(n-p, n-p)$-forms defined by algebraic $p$-pieces on $X$. Let $\mathcal{H}^{2(n-p)}(X, \mathbb{Q})$ be the set of harmonic $2(n-p)$-forms $\omega$ such that

$$\int_{\sigma} \omega \in \mathbb{Q},$$  \hfill (4)

where $\sigma$ is a singular $2(n-p)$-cycle on $X$. We first prove the following theorem.

**Theorem 2.**

$$C'_p(X) = \mathcal{H}^{n-p,n-p}(X, \mathbb{C}) \cap \mathcal{H}^{2(n-p)}(X, \mathbb{Q}).$$  \hfill (5)

Let $X$ be a smooth algebraic variety over $\mathbb{C}$.

**Definition 3.** Let $Y \subset X$. The restrictions to $Y$ of global algebraic $p$-pieces $\Gamma = \sum_i c_i \Gamma_i$ are $\Gamma \cap Y := \sum_i c_i (\Gamma_i \cap Y)$.

**Definition 4.** Let $Y \subset X$. Two restrictions $\Delta_1, \Delta_2$ to $Y$ of global algebraic $p$-pieces are equivalent if

$$\int_{\Delta_1} \eta = \int_{\Delta_2} \eta,$$  \hfill (6)

for any closed $(p, p)$-form $\eta$ on $X$. Let $C'_p(Y)$ be the set of equivalence classes defined by the restrictions to $Y$ of global algebraic $p$-pieces.

A complexified algebraic $p$-piece is a finite formal $\mathbb{C}$-linear combination $\sum_i c_i \Gamma_i (c_i \in \mathbb{C})$ of irreducible $p$-dimensional algebraic subvarieties $\{\Gamma_i\}$ on affine open subsets of $X$. 

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**Definition 5.** Let $Y \subset X$. The restrictions to $Y$ of global complexified algebraic $p$-pieces $\Gamma = \sum c_{i} \Gamma_{i}$ are $\Gamma \cap Y := \sum c_{i} (\Gamma_{i} \cap Y)$.

**Definition 6.** Let $Y \subset X$. Two restrictions $\Delta_{1}, \Delta_{2}$ to $Y$ of global complexified algebraic $p$-pieces are equivalent if

$$\int_{\Delta_{1}} \eta = \int_{\Delta_{2}} \eta.$$  

(7)

for any closed $(p, p)$-form $\eta$ on $X$. Let $C_{p}^{r}(Y) \otimes \mathbb{C}$ be the set of equivalence classes defined by the restrictions to $Y$ of global complexified algebraic $p$-pieces.

Let $1 \leq p \leq n - 1$ be an integer. Let $B_{r}(x)$ denote a closed ball with boundary in some coordinate and $\mathcal{U}$ be the set of such balls, that is, $U \in \mathcal{U}$ if there exist $z_{1} \in X$ and a coordinate $x_{1}$ around $z_{1}$ such that $U = \{||x_{1}| \leq r_{1}\}$ for some $r_{1} > 0$. Let

$$\mathcal{U}|_{B_{r}(x)^{o}} := \{U \subset B_{r}(x)^{o} \mid x \in U^{o}, U \in \mathcal{U}\}.  \tag{8}$$

**Definition 7.** Two restrictions $\omega_{1}, \omega_{2}$ to $B_{r}(x)^{o}$ of global harmonic forms are equivalent if

$$\int_{B_{r}(x)^{o}} \omega_{1} \wedge \eta = \int_{B_{r}(x)^{o}} \omega_{2} \wedge \eta$$

(9)

for any closed form $\eta$ on $X$. Let $\mathcal{H}_{p^{*}}(B_{r}(x)^{o}, \mathbb{C})$ be the set of equivalence classes of the restrictions to $B_{r}(x)^{o}$ of global harmonic $(*,*)$-forms.

**Lemma 8.** Let $x \in X$. There exists a closed ball $B_{r}(x)$ of center $x \in X$ and sufficiently small radius $r > 0$ such that for any point $z_{0} \in B_{r}(x)^{o}$ and for some coordinate of $B_{r}(x)^{o}$ there exist an $(n - p)$-dimensional complex linear subspace $M$ in $B_{r}(x)^{o}$ through the origin and a $p$-dimensional complex linear subspace $L$ in $B_{r}(x)^{o}$ orthogonal to $M$ through $z_{0}$.

**Proof.** Take an affine open neighbourhood $V$ of $x$ in $X$. Embed $V \subset \mathbb{C}^{N}$ ($N >> 0$) and consider a closed ball of center $x \in V \subset \mathbb{C}^{N}$ and sufficiently small radius $r > 0$ in $\mathbb{C}^{N}$. The intersection of the ball and $V$ is a closed ball $B_{r}(x)$ in $V$ of center $x \in V$ and radius $r$. Consider hyperplanes $H_{1}, \ldots, H_{n-p}$ through $z_{0}$ in $\mathbb{C}^{N}$. Then since $r > 0$ is sufficiently small there exist such hyperplanes such that $H_{1} \cap \cdots \cap H_{n-p} \cap V$ is a global algebraic $p$-piece on $X$ and does not go through $x$ and such that $(H_{1} \cap \cdots \cap H_{n-p} \cap V) \cap B_{r}(x)^{o}$ is a manifold. Take such global algebraic $p$-piece of the form $H_{1} \cap \cdots \cap H_{n-p} \cap V$ as $L$. The remaining statement is easy. The assertion follows. □

**Lemma 9.** $C_{n-p}(B_{r}(x)^{o}) \otimes \mathbb{C} \subset \mathcal{H}_{p^{*}}(B_{r}(x)^{o}, \mathbb{C})$.

**Proof.** Let $\Gamma \in C_{n-p}(B_{r}(x)^{o}) \otimes \mathbb{C}$. By Theorem 1 applied on $2n$-spheres and an easy argument there exists a global harmonic $(s, s)$-form $\Omega$ on $X$ such that

$$\int_{\Gamma \cap B_{r}(x)^{o}} \Xi = \int_{B_{r}(x)^{o}} \Omega \wedge \Xi$$

(10)
for any closed \((n-s,n-s)\)-form \(\Xi\) on \(X\). Thus \([\Gamma]|_{B_r(x)^o} = [\Omega]|_{B_r(x)^o}\) and the assertion follows.

Let \([\Gamma] \in C'_p(B_r(x)^o) \otimes \mathbb{C}\) and \([\eta] \in \mathscr{H}^{p,p}(B_r(x)^o, \mathbb{C})\). Let
\[
\int_{\Gamma}[\eta] := \{(U, \frac{1}{|U|} \int_U [\Gamma]|u \wedge [\eta]|u)\}_{U \in \mathscr{U}|_{B_r(x)^o}}.
\]

(11)

**Lemma 10.** There exists a closed ball \(B_r(x)\) of center \(x \in X\) and sufficiently small radius \(r > 0\) such that the map
\[
C'_p(B_r(x)^o) \otimes \mathbb{C} \times \mathscr{H}^{p,p}(B_r(x)^o, \mathbb{C}) \to \text{Map}(\mathscr{U}|_{B_r(x)^o}, \mathbb{C})
\]
\[
([\Gamma], [\eta]) \mapsto \int_{\Gamma}[\eta],
\]

(12)

is a nondegenerate bilinear map. Further it suffices to consider \([\Gamma]\) defined by the restrictions of \(p\)-algebraic \(p\)-pieces of which components are smooth on \(B_r(x)^o\).

**Proof.** By Lemma 8 there exists a closed ball \(B_r(x)\) of center \(x \in X\) and sufficiently small radius \(r > 0\) such that for any point \(z_0 \in B_r(x)^o\) and for some coordinate of \(B_r(x)^o\) there exist an \((n-p)\)-dimensional complex linear subspace \(M\) in \(B_r(x)^o\) through the origin and a \(p\)-dimensional complex linear subspace \(L\) in \(B_r(x)^o\) orthogonal to \(M\) through \(z_0\). Let \([\eta] \in \mathscr{H}^{p,p}(B_r(x)^o, \mathbb{C})\). Let \(z_0 \in B_r(x)^o\). Divide \(\eta = \alpha_L + \beta\), where \(\eta|_{\mathbb{C} \otimes L}(z_0') = \alpha_L(z_0')\) for any \(z_0'\) near \(z_0\). Assume \(\alpha_L(z_0) \neq 0\) then it is obvious that
\[
\int_{L \cap B_{\epsilon}(z_0)} \eta \neq 0 \quad \text{(}\epsilon > 0\text{ is small}).
\]

(14)

This contradiction shows \(\alpha_L(z_0) = 0\) and \(\eta(z_0) = \beta(z_0)\). Change coordinates and consider all such \(L\) (cf. the proof of Lemma 8). Combining the resulting formulas it is, by an elementary argument of exterior algebra, obtained that \(\eta(z_0) = 0\). Since \(z_0 \in B_r(x)^o\) is arbitrary it follows that \([\eta] = 0\). Now it is proved that
\[
\int_{\Gamma}[\eta] = \{(U, \frac{1}{|U|} \int_U [\Gamma]|u \wedge [\eta]|u)\}_{U \in \mathscr{U}|_{B_r(x)^o}} = \{(U, 0)\}_{U \in \mathscr{U}|_{B_r(x)^o}}
\]
\[
(\forall [\Gamma] \in C'_p(B_r(x)^o) \otimes \mathbb{C})
\]
\[
\Rightarrow [\eta] = 0.
\]

(15)

(16)

(17)

By Lemma 9 \(C'_{n-p}(B_r(x)^o) \otimes \mathbb{C} \subset \mathscr{H}^{p,p}(B_r(x)^o, \mathbb{C})\) and \(C'_p(B_r(x)^o) \otimes \mathbb{C} \subset \mathscr{H}^{n-p,n-p}(B_r(x)^o, \mathbb{C})\). Thus reversing the roles and considering \([\Gamma]\) corresponding to the restriction to \(B_r(x)^o\) of a global harmonic form as an element of
\(\mathcal{H}^{n-p,n-p}(B_r(x)^o, \mathbb{C})\) it follows that

\[
\int_{\Gamma} [\eta] = \{(U, 1/|U| \int_U [\eta]|u \wedge [\eta]|u)\}_u \in \mathcal{W}_{[B_r(x)^o]} = \{(U, 0)\}_u \in \mathcal{W}_{[B_r(x)^o]} (18)
\]

\[(\forall [\Gamma] \in C^p_{n-p}(B_r(x)^o) \otimes \mathbb{C}) \implies |\Gamma| = 0. (19)\]

The above two show the map (12)-(13) is a nondegenerate bilinear map. Further it suffices to consider \([\Gamma]\) defined by the restrictions of global algebraic \(p\)-pieces of which components are smooth on \(B_r(x)^o\). The assertion follows. \(\square\)

Let \(\omega\) be a global harmonic \((n-p,n-p)\)-form on \(X\) such that \([\omega] \in \mathcal{H}^{n-p,n-p}(X, \mathbb{C})\).

**Definition 11.** The tangent space of a \(C^1\)-manifold \(\Delta\) on an affine open subset at \(x \in \Delta\) is denoted by \(T_x\Delta\). Two finite formal \(\mathbb{C}\)-combinations \(\sum c_i \Gamma_i, c_i, c_i' \in \mathbb{C}\) of \(C^1\)-manifolds \(\{\Gamma_i\}, \{\Gamma_i'\}\) on affine open subsets of \(X\) intersect transversally if \(T_x\Gamma_i \not\subset T_x\Gamma_i'\) and \(T_x\Gamma_i \not\supset T_x\Gamma_i'\) for any \(l, l'\) and for any \(x \in \Gamma_i \cap \Gamma_i'\).

**Lemma 12.** There exist a cover \(\{U_\lambda\}_{\lambda \in \Lambda}\) of \(X\) consisting of closed balls with boundary and \([\Gamma_\lambda] \in C_p(U_\lambda) \otimes \mathbb{C} (\lambda \in \Lambda)\) defined by the restriction of a global algebraic \(p\)-piece of which components are smooth on \(U_\lambda\) satisfying the following: (i) for any global harmonic \((p,p)\)-form \(\eta\) on \(X\)

\[
\int_{\Gamma_\lambda} [\eta]|u_\lambda = \int_{U_\lambda} [\omega]|u_\lambda \wedge [\eta]|u_\lambda, (21)
\]

and (ii) if \(U_\lambda \cap U_\mu \not= \emptyset\) then for any global harmonic \((p,p)\)-form \(\eta\) on \(X\)

\[
\int_{\Gamma_\lambda \cap U_\mu} [\eta]|u_\lambda \cap u_\mu = \int_{\Gamma_\lambda \cap U_\mu} [\eta]|u_\lambda \cap u_\mu. (22)
\]

Furthermore the above are taken so that \(\Gamma_\lambda\) and \(\partial U_\lambda\) intersect transversally.

**Proof.** Since \(C_p^p(B_r(x)^o) \otimes \mathbb{C} \subset \mathcal{H}^{n-p,n-p}(B_r(x)^o, \mathbb{C})\) there exists a pairing

\[
\mathcal{H}^{n-p,n-p}(B_r(x)^o, \mathbb{C}) \times \mathcal{H}^{p,p}(B_r(x)^o, \mathbb{C}) \rightarrow \text{Map}(\mathcal{W}_{[B_r(x)^o]}, \mathbb{C}) (23)
\]

extending the map (12)-(13). By Lemma 10 this new pairing is also nondegenerate. Observe that the old pairing is nondegenerate. Thus it follows by linear algebra that \(C_p^p(B_r(x)^o) \otimes \mathbb{C} = \mathcal{H}^{n-p,n-p}(B_r(x)^o, \mathbb{C})\).

From above there exists \([\Gamma_\lambda] \in C_p^p(B_r(x)^o)\) for each \(x \in X\) such that for any \(U \in \mathcal{W}_{[B_r(x)^o]}\)

\[
\int_{U} [\Gamma_\lambda]|u \wedge [\eta]|u = \int_{U} [\omega]|u \wedge [\eta]|u \quad (\forall [\eta] \in \mathcal{H}^{p,p}(B_r(x)^o, \mathbb{C})), (24)
\]
In particular $[\Gamma_x]$ is such that

$$
\int_{B_{r'(x)}} [\Gamma_x] |_{B_{r}(x)} \wedge [\eta] |_{B_{r}(x)} = \int_{B_{r'(x)}} [\omega] |_{B_{r}(x)} \wedge [\eta] |_{B_{r}(x)} \quad (\forall [\eta] \in \mathcal{H}_p^p(B_r(x), \mathbb{C}) \ (0 < r' << r)).
$$

(26)

Further $[\Gamma_x]$ is taken to be an equivalence class defined by the restriction of a global algebraic $p$-piece of which components are smooth on $B_r(x)$. Thus there exist a cover $\{U_\lambda\}_{\lambda \in \Lambda}$ consisting of closed balls with boundary and $[\Gamma_\lambda] \in \mathcal{C}_p(U_\lambda) \otimes \mathbb{C}$ $(\lambda \in \Lambda)$ defined by the restriction of a global algebraic $p$-piece of which components are smooth on $U_\lambda$ satisfying the following: (i) for any global harmonic $(p, p)$-form $\eta$ on $X$

$$
\int_{\Gamma_\lambda} [\eta] |_{U_\lambda} = \int_{U_\lambda} [\omega] |_{U_\lambda} \wedge [\eta] |_{U_\lambda}
$$

(28)

and (ii) if $U_\lambda \cap U_\mu \neq \emptyset$ then for any global harmonic $(p, p)$-form $\eta$ on $X$

$$
\int_{\Gamma_\lambda \cap U_\mu} [\eta] |_{U_\lambda \cap U_\mu} = \int_{\Gamma_\mu \cap U_\lambda} [\eta] |_{U_\mu \cap U_\lambda}.
$$

(29)

Since $\Gamma_\lambda$’s are smooth, by shrinking $U_\lambda$’s, the above are taken so that $\Gamma_\lambda$ and $\partial U_\lambda$ intersect transversally. The assertion follows.

The following lemma is a special case of [3], Chapter I, Section 5.F, (5.26) Corollary.

**Lemma 13.** Let $M$ be a connected smooth complex variety. Let $A \subset M$ be a complex analytic subvariety of $M$. Assume $\dim A < \dim M$. Then $M \setminus A$ is connected.

**Lemma 14.** There exists an equivalence class defined by a global complexified algebraic $p$-piece $\Gamma$ of which components are smooth on $X$ such that $(|\Gamma| - [\omega]) = 0$ on $X$.

**Proof.** Define $\Gamma := \Gamma_\lambda$ on $U_\lambda$ and then $[\Gamma] |_{U_\lambda} = [\Gamma_\lambda] = [\omega] |_{U_\lambda}$. We note that $\Gamma_\lambda$ is taken so that each component of $\Gamma_\lambda$ is smooth on $U_\lambda$ and that $\Gamma_\lambda$ intersects with $\partial U_\lambda$ transversally.

The restrictions $\Gamma_\lambda$ of global complexified algebraic $p$-pieces correspond to those $\Phi_\lambda$ of global harmonic $(n-p, n-p)$-forms. By construction $[\Phi_\lambda] - [\omega] |_{U_\lambda} = 0$. Define

$$
\Phi := \Phi_\lambda \ (\text{on} \ U_\lambda).
$$

(30)

Observe that the RHS is well-defined. Let $\gamma$ be a complex analytic variety
appearing in $\Gamma_\lambda \cap (U_\lambda \cap U_{\lambda'}) - \Gamma_{\lambda'} \cap (U_\lambda \cap U_{\lambda'})$. Observe that

$$\int_{\partial(U_\lambda \cap U_{\lambda'})} ([\Phi_\lambda]|_{U_\lambda \cap U_{\lambda'}}) \wedge \theta$$

$$= \int_{\partial(U_\lambda \cap U_{\lambda'})} 0 \wedge \theta$$

$$= 0$$

for any $(2p-1)$-form $\theta$ and $\gamma \cap \partial(U_\lambda \cap U_{\lambda'})$ is of measure 0 on $\partial(U_\lambda \cap U_{\lambda'})$.

Observe that each component of $\Gamma_\lambda$ (resp. $\Gamma_{\lambda'}$) is smooth on $U_\lambda$ (resp. $U_{\lambda'}$). Recall that, for any $\mu$, $\Gamma_\mu$ is taken so that $\Gamma_\mu$ intersects with $\partial U_\mu$ transversally (see Lemma 12). If one component $\delta_\lambda$ of $\Gamma_\lambda$ and one component $\delta_{\lambda'}$ of $\Gamma_{\lambda'}$ are such that $T_x \delta_\lambda \neq T_x \delta_{\lambda'}$ for some $x \in \delta_\lambda \cap \delta_{\lambda'} \cap \partial(U_\lambda \cap U_{\lambda'})$ then $\dim(\delta_\lambda \cap \delta_{\lambda'}) < p$. On the other hand $\delta_\lambda \cap \partial U_\lambda$ and $\delta_{\lambda'} \cap \partial U_{\lambda'}$ are empty or of real dimension $(2p-1)$. Thus the above measure property shows a contradiction and the union of the tangent spaces (and the base spaces) of components of $\Gamma_\lambda$ on $\partial(U_\lambda \cap U_{\lambda'})$ and that of those of components of $\Gamma_{\lambda'}$ on $\partial(U_\lambda \cap U_{\lambda'})$ coincide. Further, from this, the above measure property and the definition of $\Phi_\lambda$ show that the coefficients of $\Gamma_\lambda \cap (U_\lambda \cap U_{\lambda'})$ and $\Gamma_{\lambda'} \cap (U_\lambda \cap U_{\lambda'})$ coincide. Let $\Gamma := \Gamma_\lambda$ (on $U_\lambda$).

$\Gamma$ intersects with $\partial(U_\lambda) \cap U_{\lambda'}$ transversally so that any component of $\Gamma$ is locally a graph of $C^1$-functions that are holomorphic outside a proper smooth manifold. By taking limit it is obvious that the functions satisfy Cauchy-Riemann equations and hence any component of $\Gamma$ is smooth complex analytic on $\bigcup_{\lambda \in \Lambda} U_\lambda = X$.

Let $\gamma$ be a component of $\Gamma$. By definition $\gamma$ may be taken to be a smooth complex analytic variety such that for any $z_0 \in X$ there exists an open ball $U_{z_0}^\circ$ around $z_0$ and polynomials $P_{U_{z_0}^\circ \cap U_{z_0}^\circ 1}, \ldots, P_{U_{z_0}^\circ \cap U_{z_0}^\circ}$ defining $\gamma$ on $U_{z_0}^\circ$. Define a set $V$ in an affine open subset $S$ containing $U_{z_0}^\circ$ by

$$V := \{P_{U_{z_0}^\circ 1}(z) = \cdots = P_{U_{z_0}^\circ \cap U_{z_0}^\circ}(z) = 0\}.$$  

Assume $\gamma \cap S \not\subseteq V$. Then $(\gamma \cap S) \cap V \not\subseteq \gamma \cap S$. On the other hand $\gamma$ is smooth and connected and $\dim((\gamma \cap (X \setminus S)) < p$ so that, by Lemma 13, $\gamma \cap S$ is smooth and connected. From the above, however, $(\gamma \cap S) \cap V$ has a component of dimension $p$. This contradiction shows that $(\gamma \cap S) \subseteq V$. Since $z_0 \in X$ is arbitrary and any affine open subset is Noetherian, by an easy argument, it is obtained that $\gamma$ is defined by finitely many polynomials on any affine open subset. Thus $\gamma$ is an algebraic variety and since $\gamma$ is arbitrary $\Gamma$ is a complexified algebraic $p$-piece. The assertion follows.

A complexified algebraic $p$-cycle is a finite formal $\mathbb{C}$-linear combination $\sum_i c_i \Gamma_i$ ($c_i \in \mathbb{C}$) of irreducible $p$-dimensional algebraic subvarieties $\{\Gamma_i\}$ on $X$.  

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Remark 15. In Lemma 14 \( \Gamma \) may be taken to be a complexified algebraic \( p \)-cycle.

Let \( \mathcal{H}^{2p}(X, \mathbb{Q}) \) be the set of harmonic \( 2p \)-forms \( \eta \) on \( X \) such that \( \int_{\sigma} \eta \in \mathbb{Q} \), where \( \sigma \) is a singular \( 2p \)-cycle.

Theorem 16. Let \( X \) be a smooth algebraic variety over \( \mathbb{C} \). Let \( \mathcal{H}_p''(X) \) be the set of harmonic \((n-p, n-p)\)-forms defined by algebraic \( p \)-pieces on \( X \) and let \( \mathcal{H}^{p,p}(X, \mathbb{C}) \cap \mathcal{H}^{2p}(X, \mathbb{Q}) \). Then the following pairing is nondegenerate:

\[
\mathcal{H}_p''(X) \times \mathcal{H}^{p,p}(X, \mathbb{Q}) \rightarrow \text{Map}(\mathcal{U}, \mathbb{C}),
\]

\[
((\Gamma), [\eta]) \mapsto \{(U, \frac{1}{|U|} \int_U [\Gamma]|_U \wedge [\eta]|_U)\}_{U \in \mathcal{U}}.
\]

Proof. Let \( \mathcal{H}^{n-p,n-p}(X, \mathbb{Q}) \subset \mathcal{H}^{n-p,n-p}(X, \mathbb{C}) \cap \mathcal{H}^{2(n-p)}(X, \mathbb{Q}) \). Observe that by Lemma 9 \( \mathcal{H}_p''(X) \subset \mathcal{H}^{n-p,n-p}(X, \mathbb{C}) \cap \mathcal{H}^{2(n-p)}(X, \mathbb{Q}) \). If \( \eta \in \mathcal{H}^{p,p}(X, \mathbb{Q}) \) satisfies \( \int_{\Gamma \cap U} \eta = 0 \) \( (\forall U \in \mathcal{U}) \) for any \( [\Gamma] \in \mathcal{H}_p''(X) \) then, by Lemma 10, \( \eta = 0 \) on any \( U \in \mathcal{U} \). It follows that \( \eta = 0 \) on \( X \). On the other hand assume \( [\Gamma] \in \mathcal{H}_p''(X) \subset \mathcal{H}^{n-p,n-p}(X, \mathbb{Q}) \) satisfies \( \int_{\Gamma \cap U} \eta = 0 \) \( (\forall U \in \mathcal{U}) \) for any \( \eta \in \mathcal{H}^{p,p}(X, \mathbb{Q}) \). By an elementary argument it is obtained that the following pairing (which extends (36)-(37)) is nondegenerate:

\[
\mathcal{H}^{n-p,n-p}(X, \mathbb{Q}) \times \mathcal{H}^{p,p}(X, \mathbb{Q}) \rightarrow \text{Map}(\mathcal{U}, \mathbb{C}).
\]

Then \( [\Gamma] = 0 \) and the following pairing is nondegenerate:

\[
\mathcal{H}_p''(X) \times \mathcal{H}^{p,p}(X, \mathbb{Q}) \rightarrow \text{Map}(\mathcal{U}, \mathbb{C}).
\]

The assertion follows. \( \square \)

Proof of Theorem 2. By Lemma 9 \( \mathcal{H}_p''(X) \subset \mathcal{H}^{n-p,n-p}(X, \mathbb{Q}) \). Observe that the following pairing is nondegenerate:

\[
\mathcal{H}^{n-p,n-p}(X, \mathbb{Q}) \times \mathcal{H}^{p,p}(X, \mathbb{Q}) \rightarrow \text{Map}(\mathcal{U}, \mathbb{C}).
\]

On the other hand by Theorem 16 the following pairing is nondegenerate:

\[
\mathcal{H}_p''(X) \times \mathcal{H}^{p,p}(X, \mathbb{Q}) \rightarrow \text{Map}(\mathcal{U}, \mathbb{C}).
\]

Thus by linear algebra

\[
\mathcal{H}_p''(X) = \mathcal{H}^{n-p,n-p}(X, \mathbb{Q}) = \mathcal{H}^{n-p,n-p}(X, \mathbb{C}) \cap \mathcal{H}^{2(n-p)}(X, \mathbb{Q}) \).
\]

The assertion follows. \( \square \)

An algebraic \( p \)-cycle is a finite formal \( \mathbb{Q} \)-linear combination \( \sum_i c_i \Gamma_i \) \((c_i \in \mathbb{Q})\) of irreducible \( p \)-dimensional algebraic subvarieties \( \{\Gamma_i\} \) on \( X \).
**Definition 17.** Two algebraic p-cycles $\Delta_1, \Delta_2$ are equivalent if

$$ \int_{\Delta_1} \eta = \int_{\Delta_2} \eta. $$

(43)

for any closed $(p, p)$-form $\eta$ on $X$. Let $C_p(X)$ be the set of equivalence classes of algebraic p-cycles on $X$.

By Remark 15, the set $C'_p(X)$ may be replaced with the set $C_p(X)$, i.e. the following theorem holds.

**Theorem 18 (Mixed Hodge Conjecture).**

$$ C_p(X) = \mathcal{H}^{n-p,n-p}(X, \mathbb{C}) \cap \mathcal{H}^{2(n-p)}(X, \mathbb{Q}). $$

(44)

4 Mixed Motives

**Definition 19.** Let $\mathcal{C}$ be the category of smooth algebraic varieties over $\mathbb{C}$ and $\mathcal{D}_\mathbb{Q}$ the category of bigraded $\mathbb{Q}$-vector spaces. A mixed Weil cohomology theory is a contravariant functor $H^{*,*}: \mathcal{C} \to \mathcal{D}_\mathbb{Q}$ such that the following holds.

(i) Let $X \in \mathcal{C}$ be of dimension $n$ then $H^{p,q}(X) = 0$ unless $0 \leq p, q \leq n$.

(ii) Let $X \in \mathcal{C}$ be of dimension $n$ then the following mixed Serre duality holds, i.e. the following is a nondegenerate pairing:

$$ H^{p,q}(X) \times H^{n-p,n-q}(X) \to \text{Map}(\mathcal{H}, \mathbb{C}), $$

(45)

where $0 \leq p, q \leq n$.

(iii) The Künneth formula holds: for $X, Y \in \mathcal{C}$

$$ H^{*,*}(X \times Y) \simeq H^{*,*}(X) \otimes H^{*,*}(Y). $$

(46)

(iv) Let $X \in \mathcal{C}$ and $Z_p(X)$ the set of algebraic p-cycles on $X$. There exists a cycle map $\gamma_X: Z_p(X) \to H^{n-p,n-p}(X)$.

(v) Let $X \in \mathcal{C}$ be of dimension $n$. Then there exists an isomorphism $H^{n,n}(X) \simeq \mathcal{H}^{n,n}(X, \mathbb{Q})$ compatible with the cycle maps, where $\mathcal{H}^{n,n}(X, \mathbb{Q})$ is the set of harmonic $(n, n)$-forms $\omega$ on $X$ such that $\int_\sigma \omega \in \mathbb{Q}$, where $\sigma$ is a singular $2n$-cycle on $X$.

(vi) Let $X \in \mathcal{P}(\mathbb{C})$. Then $H^{p, p}(X) \subset \mathcal{H}^{p, p}(X, \mathbb{Q})$.

**Theorem 20.** Let $X \in \mathcal{C}$ be of dimension $n$. Mixed Weil cohomology groups $H^{*,*}(X)$ are isomorphic to $\mathcal{H}^{*,*}(X, \mathbb{Q})$.

**Proof.** Let $\Gamma$ be an algebraic p-cycle. If $[\Gamma] \neq 0$, then $0 \neq [\Gamma] \cup *[\Gamma]$ and by Definition 19, (v), the homomorphism $\gamma_X(\Gamma) \cup (*\gamma_X(\Gamma)) \to [\Gamma] \cup *[\Gamma]$, where $*\gamma_X(\Gamma)$ is the mixed Serre dual of $\gamma_X(\Gamma)$, is well-defined. Especially $[\Gamma] \neq 0$ implies $\gamma_X(\Gamma) \neq 0$. Thus $\gamma_X(\Gamma) = 0$ implies $[\Gamma] = 0$. Conversely reversing the roles it follows that $[\Gamma] = 0$ implies $\gamma_X(\Gamma) = 0$. 

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Let $Y \in \mathcal{C}$. By Definition 19 (iv) and Theorem 18, from above, it is obtained that \( \mathcal{H}^{p,p}(Y, \mathbb{Q}) \subset H^{p,p}(Y) \) and by Definition 19, (vi) it is obtained that \( H^{p,p}(Y) \subset \mathcal{H}^{p,p}(Y, \mathbb{Q}) \). Thus \( H^{p,p}(Y) \cong \mathcal{H}^{p,p}(Y, \mathbb{Q}) \).

Let $G := \{m_1 + im_2 \mid m_1, m_2 \in \mathbb{Z}\}$ and $T := \mathbb{C}/G$. By Definition 19, (v) and (ii) it is obtained that \( H^{p,p}(T) \cong \mathcal{H}^{p,p}(T, \mathbb{Q}) \) \((p = 0, 1)\). Consider \( H^{1,1}(T^2)(\cong \mathcal{H}^{1,1}(T^2, \mathbb{Q})) \). Observe that \( \mathcal{H}^{0,0}(T, \mathbb{Q}), \mathcal{H}^{1,1}(T, \mathbb{Q}) \) and \( \mathcal{H}^{1,1}(T^2, \mathbb{Q}) \) are finite dimensional. By Definition 19, (iii) and (ii) and by an elementary argument it is obtained that \( H^{1,0}(T) \cong \mathcal{H}^{1,0}(T, \mathbb{Q}) \) and \( H^{0,1}(T) \cong \mathcal{H}^{0,1}(T, \mathbb{Q}) \). By Definition 19, (i) and (iii) it follows that \( H^{p,q}(T^n) \cong \mathcal{H}^{p,q}(T^n, \mathbb{Q}) \).

Observe that
\[
0 < \text{dim}_\mathbb{Q} \mathcal{H}^{p,q}(T^n, \mathbb{Q}) < \infty \quad (0 \leq p, q \leq n),
\]
and that
\[
H^{p,p}(X \times T^n) = \bigoplus_{\substack{p_1+q_1 = p \\ q_1+q_2 = p}} H^{p_1,q_1}(X) \otimes H^{p_2,q_2}(T) \quad (0 \leq p \leq n).
\]

Thus it follows by induction that \( H^{p,q}(X) \cong \mathcal{H}^{p,q}(X, \mathbb{Q}) \) \((0 \leq p, q \leq n)\). The assertion follows.

\begin{proof}
\end{proof}

**Definition 21.** Let $X$ be an algebraic variety over $\mathbb{C}$. Let $\tilde{X} \to X$ be its resolution of singularity. The mixed motive of $X$ of degree $k$ \((k \in \mathbb{Z})\) is the set \( \mathcal{H}^k(\tilde{X}, \mathbb{Q}) \) of harmonic $k$-forms $\omega$ on $X$ such that $\int_\sigma \omega \in \mathbb{Q}$, where $\sigma$ is a singular $k$-cycle on $X$.

**References**


