On the Definition of Motives on Algebraic Varieties

Hideto Ishihara

October 10, 2017

e-mail: h.isihara26@gmail.com

Abstract

Let $X$ be an algebraic variety over $\mathbb{C}$. Let $\tilde{X} \to X$ be its resolution of singularity. Then the $L^2$-motive of $X$ of degree $k$ ($k \in \mathbb{Z}$) is the set of harmonic $k$-forms $\omega$ on $\tilde{X}$ such that $\int_\sigma \omega \in \mathbb{Q}$, where $\sigma$ is a singular $k$-$L^2$-cycle of $\tilde{X}$.

1 Introduction

In [4] Grothendieck proposed the concept of motives, 'a decomposition of an algebraic variety over $\mathbb{C}$ into equidimensional pieces.' Let $X$ be an algebraic variety over $\mathbb{C}$ and $\tilde{X} \to X$ its resolution of singularity. With the use of $L^2$-Hodge Conjecture the axioms of $L^2$-Weil cohomology theory are formulated. Then we define the $L^2$-motive of degree $k$ ($k \in \mathbb{Z}$) as the set $\mathcal{H}^k(X, \mathbb{Q})$ of harmonic $k$-forms $\omega$ on $\tilde{X}$ such that $\int_\sigma \omega \in \mathbb{Q}$, where $\sigma$ is a singular $k$-$L^2$-cycle of $\tilde{X}$.

2 $L^2$-Hodge Conjecture

Let $X$ be a smooth algebraic variety over $\mathbb{C}$. By definition the set $\mathcal{H}^k(X, \mathbb{C})$ of harmonic $k$-forms on $X$ decomposes as follows.

$$\mathcal{H}^k(X, \mathbb{C}) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X, \mathbb{C}),$$

where $\mathcal{H}^{p,q}(X, \mathbb{C})$ is the set of harmonic $(p, q)$-forms on $X$. Let $0 \leq p \leq n := \dim X$ be an integer. An algebraic $p$-piece is a finite formal $\mathbb{Q}$-linear combination $\sum_l c_l \Gamma_l$ ($c_l \in \mathbb{Q}$) of irreducible $p$-dimensional algebraic subvarieties $\{\Gamma_l\}$ on affine open sets of $X$. Let $C'_p(X)$ be the set of harmonic $(n-p, n-p)$-forms
defined by algebraic $p$-pieces on $X$. Let $\mathcal{H}^{2(n-p)}(X, \mathbb{Q})$ be the set of harmonic $2(n-p)$-forms $\omega$ such that
\[
\int_{\sigma} \omega \in \mathbb{Q},
\]
where $\sigma$ is a singular $2(n-p)$-$L^2$-cycle on $X$.

**Theorem 1.**

\[
C'_p(X) = \mathcal{H}^{n-p,n-p}(X, \mathbb{C}) \cap \mathcal{H}^{2(n-p)}(X, \mathbb{Q}).
\]

Let $X$ be a smooth algebraic variety over $\mathbb{C}$.

**Definition 2.** Let $Y \subset X$. The restrictions to $Y$ of global algebraic $p$-pieces $\Gamma = \sum_i c_i \Gamma_i$ are $\Gamma \cap Y := \sum_i c_i (\Gamma_i \cap Y)$.

**Definition 3.** Let $Y \subset X$. Two restrictions $\Delta_1, \Delta_2$ to $Y$ of global algebraic $p$-pieces are equivalent if
\[
\int_{\Delta_1} \eta = \int_{\Delta_2} \eta,
\]
for any closed $(p,p)$-$L^2$-form $\eta$ on $X$. Let $C'_p(Y)$ be the set of equivalence classes defined by the restrictions to $Y$ of global algebraic $p$-pieces.

A complexified algebraic $p$-piece is a formal $\mathbb{C}$-linear combination $\sum_i c_i \Gamma_i$ ($c_i \in \mathbb{C}$) of irreducible $p$-dimensional algebraic subvarieties $\{\Gamma_i\}$ on affine open sets of $X$.

**Definition 4.** Let $Y \subset X$. The restrictions to $Y$ of global complexified algebraic $p$-pieces $\Gamma = \sum_i c_i \Gamma_i$ are $\Gamma \cap Y := \sum_i c_i (\Gamma_i \cap Y)$.

**Definition 5.** Let $Y \subset X$. Two restrictions $\Delta_1, \Delta_2$ to $Y$ of global complexified algebraic $p$-pieces are equivalent if
\[
\int_{\Delta_1} \eta = \int_{\Delta_2} \eta,
\]
for any closed $(p,p)$-$L^2$-form $\eta$ on $X$. Let $C'_p(Y) \otimes \mathbb{C}$ be the set of equivalence classes defined by the restrictions to $Y$ of global complexified algebraic $p$-pieces.

Let $1 \leq p \leq n-1$ be an integer. Let $B_r(x)$ denote a closed ball with boundary in some coordinate and $\mathcal{U}$ be the set of such balls, that is, $U \in \mathcal{U}$ if there exist $z_1 \in X$ and a coordinate $x_1$ around $z_1$ such that $U = \{ ||x_1|| \leq r_1 \}$ for some $r_1 > 0$. Let
\[
\mathcal{U}|_{B_r(x)^o} := \{ U \subset B_r(x)^o \mid x \in U^o, U \in \mathcal{U} \}.
\]
Definition 6. Two restrictions \( \omega_1, \omega_2 \) to \( B_r(x)^v \) of global harmonic forms are equivalent if

\[
\int_{B_r(x)^v} \omega_1 \wedge \eta = \int_{B_r(x)^v} \omega_2 \wedge \eta
\]

for any closed \( L^2 \)-form \( \eta \) on \( X \). Let \( \mathcal{H}^{\ast, \ast}(B_r(x)^v, \mathbb{C}) \) be the set of equivalence classes of the restrictions to \( B_r(x)^v \) of global harmonic \((\ast, \ast)\)-forms.

Lemma 7. Let \( x \in X \). There exists a closed ball \( B_r(x) \) of center \( x \in X \) and sufficiently small radius \( r > 0 \) such that for any point \( z_0 \in B_r(x)^v \) and for some coordinate of \( B_r(x)^v \) there exist an \((n-p)\)-dimensional complex linear subspace \( M \) in \( B_r(x)^v \) through the origin and a \( p \)-dimensional complex linear subspace \( L \) in \( B_r(x)^v \) orthogonal to \( M \) through \( z_0 \).

Proof. Take an affine open neighbourhood \( V \) of \( x \) in \( X \). Embed \( V \subset \mathbb{C}^N \) \((N >> 0)\) and consider a closed ball of center \( x \in V \subset \mathbb{C}^N \) and sufficiently small radius \( r > 0 \) in \( \mathbb{C}^N \). The intersection of the ball and \( V \) is a closed ball \( B_r(x) \) in \( V \) of center \( x \in V \) and radius \( r \). Consider hyperplanes \( H_1, \ldots, H_{n-p} \) through \( z_0 \) in \( \mathbb{C}^N \). Then since \( r > 0 \) is sufficiently small there exist such hyperplanes such that \( H_1 \cap \cdots \cap H_{n-p} \cap V \) is a global algebraic \( p \)-piece on \( X \) and does not go through \( x \) and such that \( (H_1 \cap \cdots \cap H_{n-p} \cap V) \cap B_r(x)^v \) is a manifold. Take such global algebraic \( p \)-pieces of the form \( H_1 \cap \cdots \cap H_{n-p} \cap V \) as \( L \). The remaining statement is easy. The assertion follows. \( \square \)

Lemma 8. \( C_{n-s}^r(B_r(x)^v) \otimes \mathbb{C} \subset \mathcal{H}^{s, s}(B_r(x)^v, \mathbb{C}) \).

Proof. Extend a complexified algebraic \((n-s)\)-piece \( \Gamma \) to \( X \) and by an elementary argument there exists a global harmonic \((s, s)\)-form \( \Omega \) on \( X \) such that

\[
\int_{\Gamma} \Xi = \int_X \Omega \wedge \Xi
\]

for any closed \((n-s, n-s)\)-\( L^2 \)-form \( \Xi \) on \( X \). Thus \( [\Gamma]_{B_r(x)^v} = [\Omega]_{B_r(x)^v} \) and the assertion follows. \( \square \)

Let \( [\Gamma] \in C_r^p(B_r(x)^v) \otimes \mathbb{C} \) and \([\eta] \in \mathcal{H}^{p, p}(B_r(x)^v, \mathbb{C})\). Let

\[
\int_{[\Gamma]} [\eta] := \{(U, \frac{1}{|U|}) \int_U [\Gamma] |U \wedge [\eta]| |U| \} \in \mathcal{H}^{p, p}(B_r(x)^v, \mathbb{C}).
\]

Lemma 9. There exists a closed ball \( B_r(x) \) of center \( x \in X \) and sufficiently small radius \( r > 0 \) such that the map

\[
C_r^p(B_r(x)^v) \otimes \mathbb{C} \times \mathcal{H}^{p, p}(B_r(x)^v, \mathbb{C}) \rightarrow \text{Map}(\mathcal{H}_{B_r(x)^v}, \mathbb{C})
\]

\[
([\Gamma], [\eta]) \mapsto \int_{[\Gamma]} [\eta],
\]

is a nondegenerate bilinear map. Further it suffices to consider \([\Gamma] \) defined by the restrictions of global algebraic \( p \)-pieces of which components are smooth on \( B_r(x)^v \).
Proof. By Lemma 7 there exists a closed ball $B_r(x)$ of center $x \in X$ and sufficiently small radius $r > 0$ such that for any point $z_0 \in B_r(x)^{\circ}$ and for some coordinate of $B_r(x)^{\circ}$ there exist an $(n-p)$-dimensional complex linear subspace $M$ in $B_r(x)^{\circ}$ through the origin and a $p$-dimensional complex linear subspace $L$ in $B_r(x)^{\circ}$ orthogonal to $M$ through $z_0$. Let $[\eta] \in \mathcal{H}_{n-p}(B_r(x)^{\circ}, \mathbb{C})$. Let $z_0 \in B_r(x)^{\circ}$. Divide $\eta = \alpha_L + \beta$, where $\eta|_{z_0} + L(z'_0) = \alpha_L(z'_0)$ for any $z'_0$ near $z_0$. Assume $\alpha_L(z_0) \neq 0$ then it is obvious that

$$\int_{L \cap B_{|\eta|+\epsilon}(x)} \eta \neq 0 \quad (\epsilon > 0 \text{ is small}).$$

(12)

This contradiction shows $\alpha_L(z_0) = 0$ and $\eta(z_0) = \beta(z_0)$. Change coordinates and consider all such $L$ (cf. the proof of Lemma 7). Combining the resulting formulas it is, by an elementary argument of exterior algebra, obtained that $\eta(z_0) = 0$. Since $z_0 \in B_r(x)^{\circ}$ is arbitrary it follows that $[\eta] = 0$. Now it is proved that

$$\int_\Gamma [\eta] = \{(U, 1/|U| \int_U [\Gamma]|U \land [\eta]|U\})U \in \mathcal{W}|_{B_r(x)^{\circ}} = \{(U, 0)\}U \in \mathcal{W}|_{B_r(x)^{\circ}}$$

(13)

$$\quad (\forall \Gamma \in C'_p(B_r(x)^{\circ}) \otimes \mathbb{C})$$

(14)

$$\Rightarrow [\eta] = 0.$$  

(15)

By Lemma 8 $C'_m(B_r(x)^{\circ}) \otimes \mathbb{C} \subset \mathcal{H}_{n-p}(B_r(x)^{\circ}, \mathbb{C})$ and $C'_p(B_r(x)^{\circ}) \otimes \mathbb{C} \subset \mathcal{H}_{n-p,n-p}(B_r(x)^{\circ}, \mathbb{C})$. Thus reversing the roles and considering $[\Gamma]$ corresponding to the restriction to $B_r(x)^{\circ}$ of a global harmonic form as an element of $\mathcal{H}_{n-p,n-p}(B_r(x)^{\circ}, \mathbb{C})$ it follows that

$$\int_\Gamma [\Gamma'] = \{(U, 1/|U| \int_U [\Gamma]|U \land [\Gamma']|U\})U \in \mathcal{W}|_{B_r(x)^{\circ}} = \{(U, 0)\}U \in \mathcal{W}|_{B_r(x)^{\circ}}$$

(16)

$$\quad (\forall \Gamma' \in C'_m(B_r(x)^{\circ}) \otimes \mathbb{C})$$

(17)

$$\Rightarrow [\Gamma] = 0.$$  

(18)

The above two show the map (10)-(11) is a nondegenerate bilinear map. Further it suffices to consider $[\Gamma]$ defined by the restrictions of global algebraic $p$-pieces of which components are smooth on $B_r(x)^{\circ}$. The assertion follows.

Let $\omega$ be a global harmonic $(n-p,n-p)$-form on $X$ such that $[\omega] \in \mathcal{H}_{n-p,n-p}(X, \mathbb{C})$.

**Definition 10.** The tangent space of a $C^1$-manifold $\Delta$ on an affine open subset at $x \in \Delta$ is denoted by $T_x \Delta$. Two finite formal $\mathbb{C}$-combinations $\sum_l c_l \Gamma_l$, $\sum_{l'} c_{l'} \Gamma'_l$ ($c_l, c_{l'} \in \mathbb{C}$) of $C^1$-manifolds $\{\Gamma_l\}$, $\{\Gamma'_l\}$ on affine open subsets of $X$ intersect transversally if $T_x \Gamma_l \not\supset T_x \Gamma'_l$ and $T_x \Gamma_l \not\supset T_x \Gamma'_l$ for any $l, l'$ and for any $x \in \Gamma_l \cap \Gamma'_l$.
Lemma 11. For any affine open subset of $X$ there exist a cover $\{U_\lambda\}_{\lambda \in \Lambda}$ of the subset consisting of closed balls with boundary and $[\Gamma_\lambda] \in C_p(U_\lambda) \otimes \mathbb{C}$ ($\lambda \in \Lambda$) defined by the restriction of a global algebraic $p$-piece of which components are smooth on $U_\lambda$ satisfying the following: (i) for any global harmonic $(p,p)$-form $\eta$ on $X$

$$\int_{\Gamma_\lambda} [\eta]|U_\lambda = \int_{U_\lambda} [\omega]|U_\lambda \wedge [\eta]|U_\lambda,$$  

and (ii) if $U_\lambda \cap U_\mu \neq \emptyset$ then for any global harmonic $(p,p)$-form $\eta$ on $X$

$$\int_{\Gamma_\lambda \cap U_\mu} [\eta]|U_\lambda \cap U_\mu = \int_{U_\mu \cap U_\lambda} [\eta]|U_\lambda \cap U_\mu.$$  

Furthermore the above are taken so that $\Gamma_\lambda$ and $\partial U_\lambda$ intersect transversally.

Proof. Since $C_p(B_r(x)^o) \otimes \mathbb{C} \subset \mathcal{H}^{n-p,n-p}(B_r(x)^o, \mathbb{C})$ there exists a pairing

$$\mathcal{H}^{n-p,n-p}(B_r(x)^o, \mathbb{C}) \times \mathcal{H}^{p,p}(B_r(x)^o, \mathbb{C}) \to \text{Map}(\mathcal{U}|_{B_r(x)^o}, \mathbb{C})$$

extending the map (10)-(11). By Lemma 9 this new pairing is also nondegenerate. Observe that the old pairing is nondegenerate and $\mathcal{H}^{n-p,n-p}(B_r(x)^o, \mathbb{C})$ and $\mathcal{H}^{p,p}(B_r(x)^o, \mathbb{C})$ are finite dimensional. Thus it follows by linear algebra that $C_p(B_r(x)^o) \otimes \mathbb{C} = \mathcal{H}^{n-p,n-p}(B_r(x)^o, \mathbb{C})$.

From above there exists $[\Gamma_x] \in C_p(B_r(x)^o)$ for each $x \in X$ such that for any $U \in \mathcal{U}|_{B_r(x)^o}$

$$\int_U [\Gamma_x]|U \wedge [\eta]|U = \int_U [\omega]|U \wedge [\eta]|U$$

\((\forall \eta \in \mathcal{H}^{p,p}(B_r(x)^o, \mathbb{C}))\),

In particular $[\Gamma_x]$ is such that

$$\int_{B_{r'}(x)} [\Gamma_x]|_{B_{r'}(x)} \wedge [\eta]|_{B_{r'}(x)} = \int_{B_{r'}(x)} [\omega]|_{B_{r'}(x)} \wedge [\eta]|_{B_{r'}(x)}$$

\((\forall \eta \in \mathcal{H}^{p,p}(B_r(x)^o, \mathbb{C}) (0 < r' < r))\). Further $[\Gamma_x]$ is taken to be an equivalence class defined by the restriction of a global algebraic $p$-piece of which components are smooth on $B_r(x)^o$. Thus since $X$ is compact there exist a finite cover $\{U_\lambda\}_{\lambda \in \Lambda}$ consisting of closed balls with boundary and $[\Gamma_\lambda] \in C_p(U_\lambda) \otimes \mathbb{C}$ ($\lambda \in \Lambda$) defined by the restriction of a global algebraic $p$-piece of which components are smooth on $U_\lambda$ satisfying the following: (i) for any global harmonic $(p,p)$-form $\eta$ on $X$

$$\int_{\Gamma_\lambda} [\eta]|U_\lambda = \int_{U_\lambda} [\omega]|U_\lambda \wedge [\eta]|U_\lambda$$

(26)
and (ii) if $U_\lambda \cap U_\mu \neq \phi$ then for any global harmonic $(p, p)$-form $\eta$ on $X$

$$\int_{\Gamma_\lambda \cap U_\mu}[\eta]|_{U_\lambda \cap U_\mu} = \int_{\Gamma_\mu \cap U_\lambda}[\eta]|_{U_\lambda \cap U_\mu}. \tag{27}$$

Since $\Gamma_\lambda$’s are smooth, by shrinking $U_\lambda$’s, the above are taken so that $\Gamma_\lambda$ and $\partial U_\lambda$ intersect transversally. The assertion follows. \hfill \Box

$\{[\Gamma_\lambda]\}_{\lambda \in \Lambda}$ defines a de Rham cohomology class $[\omega]$.

**Lemma 12.** There exists an equivalence class defined by a global complexified algebraic $p$-piece $\Gamma$ of which components are smooth on an affine open subset of $X$ such that $([\Gamma] - [\omega]) = 0$ on the subset.

**Proof.** Define $\Gamma := \Gamma_\lambda$ on $U_\lambda$ and then $[\Gamma]|_{U_\lambda} = [\Gamma_\lambda] = [\omega]|_{U_\lambda}$. We note that $\Gamma_\lambda$ is taken so that each component of $\Gamma_\lambda$ is smooth on $U_\lambda$ and that $\Gamma_\lambda$ intersects with $\partial U_\lambda$ transversally. The assertion follows.

The restrictions $\Gamma_\lambda$ of global complexified algebraic $p$-pieces correspond to those $\Phi_\lambda$ of global harmonic $(n - p, n - p)$-forms. By construction $([\Gamma] - [\omega])|_{U_\lambda} = 0$. Define

$$\Phi := \Phi_\lambda \text{ (on } U_\lambda). \tag{28}$$

We prove that the RHS is well-defined. Let $\gamma$ be a complex analytic variety appearing in $\Gamma_\lambda \cap (U_\lambda \cap U_\lambda') - \Gamma_\lambda' \cap (U_\lambda \cap U_\lambda')$. Observe that

$$\int_{\partial(U_\lambda \cap U_\lambda')} (|\Phi_\lambda| |_{U_\lambda \cap U_\lambda'} - |\Phi_\lambda'| |_{U_\lambda \cap U_\lambda'}) \wedge \theta \tag{29}$$

$$= \int_{\partial(U_\lambda \cap U_\lambda')} 0 \wedge \theta \tag{30}$$

$$= 0 \tag{31}$$

for any $(2p - 1)$-form $\theta$ and $\gamma \cap \partial(U_\lambda \cap U_\lambda')$ is of measure 0 on $\partial(U_\lambda \cap U_\lambda')$.

Observe that each component of $U_\lambda$ (resp. $\Gamma_\lambda'$) is smooth on $U_\lambda$ (resp. $U_\lambda'$). Recall that, for any $\mu$, $\Gamma_\mu$ is taken so that $\Gamma_\mu$ intersects with $\partial U_\mu$ transversally (see Lemma 11). If one component $\delta_\lambda$ of $\Gamma_\lambda$ and one component $\delta_\lambda'$ of $\Gamma_\lambda'$ are such that $T_x\delta_\lambda \neq T_x\delta_\lambda'$ for some $x \in \delta_\lambda \cap \delta_\lambda' \cap \partial(U_\lambda \cap U_\lambda')$ then $\dim(\delta_\lambda \cap \delta_\lambda') < p$. On the other hand $\delta_\lambda \cap \partial U_\lambda$ and $\delta_\lambda' \cap \partial U_\lambda'$ are empty or of real dimension $(2p - 1)$. Thus the above measure property shows a contradiction and the union of the tangent spaces (and the base spaces) of components of $\Gamma_\lambda$ on $\partial(U_\lambda \cap U_\lambda')$ and that of those of components of $\Gamma_\lambda'$ on $\partial(U_\lambda \cap U_\lambda')$ coincide. Further, from this, the above measure property shows that the coefficients of $\Gamma_\lambda \cap (U_\lambda \cap U_\lambda')$ and $\Gamma_\lambda' \cap (U_\lambda \cap U_\lambda')$ coincide. Let

$$\Gamma := \Gamma_\lambda \text{ (on } U_\lambda). \tag{32}$$
Γ intersects with \((\partial U_\lambda) \cap U_\lambda\) transversally so that any component of \(Γ\) is locally a graph of \(C^1\)-functions that are holomorphic outside a proper smooth manifold. By taking limit it is obvious that the functions satisfy Cauchy-Riemann equations and hence any component of \(Γ\) is smooth complex analytic on \(\bigcup_{\lambda \in \Lambda} U_\lambda\).

We note that \(\bigcup_{\lambda \in \Lambda} U_\lambda\) is an affine open subset of \(X\). The assertion follows. \(\square\)

Let \(\mathcal{H}^{2p}(X, \mathbb{Q})\) be the set of harmonic \(2p\)-forms \(\eta\) on \(X\) such that \(\int_\sigma \eta \in \mathbb{Q}\), where \(\sigma\) is a singular \(2p\)-\(L^2\)-cycle.

**Theorem 13.** Let \(X\) be a smooth algebraic variety over \(\mathbb{C}\). Then the set \(C^\prime_p(X)\) of harmonic \((n-p, n-p)\)-forms defined by algebraic \(p\)-pieces on \(X\) is dual to the set \(\mathcal{H}^{p,p}(X, \mathbb{C}) \cap \mathcal{H}^{2p}(X, \mathbb{Q})\).

**Proof.** Let \(\mathcal{H}^{n-p,n-p}(X, \mathbb{Q}) := \mathcal{H}^{n-p,n-p}(X, \mathbb{C}) \cap \mathcal{H}^{2(n-p)}(X, \mathbb{Q})\). Observe that by Lemma 8 \(C^\prime_p(X) \subset \mathcal{H}^{n-p,n-p}(X, \mathbb{Q})\). Observe that in Lemma 12 the affine open subset of \(X\) is taken arbitrarily. Thus if \(\eta \in \mathcal{H}^{p,p}(X, \mathbb{Q})\) satisfies \(\int_\Gamma \eta = 0\) for any \(\Gamma \in C^\prime_p(X)\) then \(\eta = 0\) on any affine open subset of \(X\). It follows that \(\eta = 0\) on \(X\). On the other hand assume \(\Gamma \in C^\prime_p(X) \subset \mathcal{H}^{n-p,n-p}(X, \mathbb{Q})\) satisfies \(\int_\Gamma \eta = 0\) for any \(\eta \in \mathcal{H}^{p,p}(X, \mathbb{Q})\). By Lemma 8, Lemma 12 and an elementary argument it is obtained that \(\mathcal{H}^{n-p,n-p}(X, \mathbb{Q})\) is dual to \(\mathcal{H}^{p,p}(X, \mathbb{Q})\). Then \(\Gamma = 0\) and \(C^\prime_p(X)\) is dual to \(\mathcal{H}^{p,p}(X, \mathbb{Q})\). The assertion follows. \(\square\)

**Proof of Theorem 1.** By Lemma 8 \(C^\prime_p(X) \subset \mathcal{H}^{n-p,n-p}(X, \mathbb{Q})\). Observe that \(\mathcal{H}^{n-p,n-p}(X, \mathbb{Q})\) is dual to \(\mathcal{H}^{p,p}(X, \mathbb{Q})\). On the other hand by Theorem 13 \(C^\prime_p(X)\) is dual to \(\mathcal{H}^{p,p}(X, \mathbb{Q})\). Thus by linear algebra

\[
C^\prime_p(X) = \mathcal{H}^{n-p,n-p}(X, \mathbb{Q})(= \mathcal{H}^{n-p,n-p}(X, \mathbb{C}) \cap \mathcal{H}^{2(n-p)}(X, \mathbb{Q})).
\] (33)

The assertion follows. \(\square\)

**Definition 14.** Let \(\mathcal{C}\) be the category of smooth algebraic varieties over \(\mathbb{C}\) and \(\mathcal{D}_\mathbb{Q}\) the category of bigraded \(\mathbb{Q}\)-vector spaces. An \(L^2\)-Weil cohomology theory is a contravariant functor \(H^* : \mathcal{C} \to \mathcal{D}_\mathbb{Q}\) such that the following holds.

1. Let \(X \in \mathcal{C}\) be of dimension \(n\) then \(H^{p,q}(X) = 0\) unless \(0 \leq p, q \leq n\).
2. Let \(X \in \mathcal{C}\) be of dimension \(n\) then the Poincaré duality holds, i.e. the following is a nondegenerate pairing:

\[
H^{p,q}(X) \times H^{n-p,n-q}(X) \to \mathbb{Q},
\] (34)

where \(0 \leq p, q \leq n\).

3. The Künneth formula holds: for \(X, Y \in \mathcal{C}\)

\[
H^{*,*}(X \times Y) \simeq H^{*,*}(X) \otimes H^{*,*}(Y).
\] (35)

4. Let \(X \in \mathcal{C}\) and \(Z^\prime_p(X)\) the set of algebraic \(p\)-pieces on \(X\). There exists a piece map \(\gamma^\prime_X : Z^\prime_p(X) \to H^{2p}(X)\).
(5) Let $X \in C$ be of dimension $n$. Then there exists an isomorphism $H^{2n}(X) \cong \mathcal{H}^{2n}(X, \mathbb{Q})$ compatible with the piece maps, where $\mathcal{H}^{2n}(X, \mathbb{Q})$ is the set of harmonic $2n$-forms $\omega$ on $X$ such that $\int_\sigma \omega \in \mathbb{Q}$, where $\sigma$ is a singular $2n$-$L^2$-cycle on $\hat{X}$.

(6) Let $X \in \mathcal{P}(\mathbb{C})$. Then $H^{p,p}(X) \subset \mathcal{H}^{p,p}(X, \mathbb{Q})$.

**Theorem 15.** Let $X \in C$ be of dimension $n$. $L^2$-Weil cohomology groups $H^{*,*}(X)$ are isomorphic to $\mathcal{H}^{*,*}(X, \mathbb{Q})$.

**Proof.** Let $\Gamma$ be an algebraic $p$-piece. If $[\Gamma] \neq 0$, then $0 < [\Gamma] \cup *[\Gamma]$ and by Definition 14, (5), the homomorphism $\gamma_X(\Gamma) \cup (\gamma_X(\Gamma)) \rightarrow [\Gamma] \cup *[\Gamma]$, where $*\gamma_X(\Gamma)$ is the Poincaré dual of $\gamma_X(\Gamma)$, is well-defined. Especially $[\Gamma] \neq 0$ implies $\gamma_X(\Gamma) \neq 0$. Thus $\gamma_X(\Gamma) = 0$ implies $[\Gamma] = 0$. Conversely reversing the roles it follows that $[\Gamma] = 0$ implies $\gamma_X(\Gamma) = 0$.

Let $Y \in C$. By Definition 14 (4) and Theorem 1, from above, it is obtained that $\mathcal{H}^{p,p}(Y, \mathbb{Q}) \subset H^{p,p}(Y)$ and by Definition 14, (6) it is obtained that $H^{p,p}(Y) \subset \mathcal{H}^{p,p}(Y, \mathbb{Q})$. Thus $H^{p,p}(Y) \simeq \mathcal{H}^{p,p}(Y, \mathbb{Q})$.

Let $G := \{m_1 + im_2 \mid m_1, m_2 \in \mathbb{Z}\}$ and $T := \mathbb{C}/G$. By Definition 14, (5) and (2) it is obtained that $H^{p,p}(T) \simeq \mathcal{H}^{p,p}(T, \mathbb{Q})$ ($p = 0, 1$). Consider $H^{1,1}(T^2)$. By Definition 14, (3) and (2) and by an elementary argument it is obtained that $H^{1,0}(T) \simeq \mathcal{H}^{1,0}(T, \mathbb{Q})$ and $H^{0,1}(T) \simeq \mathcal{H}^{0,1}(T, \mathbb{Q})$. By Definition 14, (3) it follows that $H^{p,q}(T^n) \simeq \mathcal{H}^{p,q}(T^n, \mathbb{Q})$.

Observe that
\begin{equation}
0 < \dim_{\mathbb{Q}} \mathcal{H}^{p,q}(T^n, \mathbb{Q}) < \infty \quad (0 \leq p, q \leq n),
\end{equation}
and that
\begin{equation}
H^{p,p}(X \times T^n) = \bigoplus_{p_1 + p_2 = p} H^{p_1,q_1}(X) \otimes H^{p_2,q_2}(T) \quad (0 \leq p \leq n).
\end{equation}

Thus it follows by induction that $H^{p,q}(X) \simeq \mathcal{H}^{p,q}(X, \mathbb{Q})$ ($0 \leq p, q \leq n$). The assertion follows.

**Definition 16.** Let $X$ be an algebraic variety over $\mathbb{C}$. Let $\hat{X} \rightarrow X$ be its resolution of singularity. The $L^2$-motive of $X$ of degree $k$ ($k \in \mathbb{Z}$) is the set $\mathcal{H}^k(\hat{X}, \mathbb{Q})$ of harmonic $k$-forms $\omega$ on $\hat{X}$ such that $\int_\sigma \omega \in \mathbb{Q}$, where $\sigma$ is a singular $k$-$L^2$-cycle on $\hat{X}$.

**References**

claymath.org/sites/default/files/hodge.pdf [Accessed: 18th January 2017]

[Accessed: 14th February 2016]
