On the Existence and Smoothness of Solutions of Incompressible Navier-Stokes Equations

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Abstract

Let \( I := [0, T] \) \((T > 0)\) be an interval. We prove the existence, smoothness and uniqueness of solutions of Navier-Stokes equations on \( I \times (\mathbb{R}^3 / \mathbb{Z}^3) \) and on \( I \times \mathbb{R}^3 \). Our proof is a new approach.

Keywords: Navier-Stokes equations, Fréchet manifolds, Sheaves

1 Introduction

We investigate the following Navier-Stokes equations:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= -u \cdot \nabla u + \nu \Delta u - \nabla p + f, \\
\nabla \cdot u &= 0.
\end{align*}
\]

(1)

The linear partial differential equations are rigorously examined (cf. [14]). As to Navier-Stokes equations the global existence of smooth solutions of Navier-
Stokes equations is a well-known problem (see e.g. [18], [5]). The following theorem, of which expression is slightly changed, is proved in [13].

Theorem 1. Let \( \nu > 0 \). Let \( f = 0 \). Let \( u_0 \in L^2(\mathbb{R}^3) \) such that \( \nabla \cdot u_0 = 0 \). Then there exists a weak solution \( u \) satisfying equation (1), i.e. for any compactly supported \( \phi \in C^2((0, \infty) \times \mathbb{R}^3)^3 \) with \( \nabla \cdot \phi = 0 \)

\[
\int_0^\infty \int_{\mathbb{R}^3} (u \cdot \frac{\partial \phi}{\partial t} + \sum_j \sum_k u_k u_j \partial_k \phi_j + u \cdot \nu \Delta \phi) dx dt = 0. \tag{2}
\]

holds. Further for \( t \in (0, \infty) \)

\[
||u(t, \cdot)||_{L^2(\mathbb{R}^3)}^2 + \nu \int_0^t ||\nabla u(s, \cdot)||_{L^2(\mathbb{R}^3)}^2 ds \leq ||u_0||_{L^2(\mathbb{R}^3)}^2, \tag{3}
\]

and

\[
||u(t, \cdot) - u_0||_{L^2(\mathbb{R}^3)} \to 0 \quad (t \to +0). \tag{4}
\]

Let \( \nu > 0 \) and \( I := [0, T] \) \( (T > 0) \). Let
\[
W_n := C^\infty(I, C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^n).
\]
Let
\[
(W_1)\nabla := \{ v \mid v = \nabla w \ (\exists w \in W_1) \}.
\]
We prove the following main theorem.

**Theorem 2.** Let \( \nu > 0 \). Let \( I := [0, T] \) \( (T > 0) \). Let \( u_0 \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3 \) such that \( \nabla \cdot u_0 = 0 \) and let \( f \in W_3 \). Then there exists a unique \((u, \nabla p) \in W_3 \times (W_1)\nabla\) such that
\[
\begin{aligned}
\frac{\partial u}{\partial t} &= -(u \cdot \nabla)u + \nu \Delta u - \nabla p + f, \\
\nabla \cdot u &= 0, \\
u_0 \mid_{t=0} &= u_0.
\end{aligned}
\]

The proof of Theorem 2 proceeds as follows.

Let
\[
\Phi : W_3 \times (W_1)\nabla \to W_3 \times (W_3)_{\text{div}} \times C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3
\]
be given by
\[
(u, P) \mapsto \begin{bmatrix}
\dot{u} + (u \cdot \nabla)u - \nu \Delta u + P \\
\nabla \cdot u \\
u(0)
\end{bmatrix},
\]
where
\[
(W_3)_{\text{div}} := \{ v \mid v = \nabla \cdot w \ (\exists w \in W_3) \}.
\]
\(\Phi\) is \(C^\infty\) in \((u, P)\) and the map
\[
d\Phi : (h, \beta) \mapsto \begin{bmatrix}
\dot{h} + (u \cdot \nabla)h + (h \cdot \nabla)u - \nu \Delta h + \beta \\
\nabla \cdot h \\
h(0)
\end{bmatrix}
\]
is a linear isomorphism from
\[
W_3 \times (W_1)\nabla
\]
to
\[
\{(a, b, c) \in W_3 \times (W_3)_{\text{div}} \times C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3 \mid \nabla \cdot c = b(0) \}.
\]
Let
\[ \mathcal{X} := W_3 \times (W_1)_{\nabla} \] (14)
and
\[ \mathcal{X} := \{ r := (f, r_2, u_0) \in W_3 \times (W_3)_{\text{div}} \times C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3 \mid \nabla \cdot u_0 = r_2(0) \}. \] (15)

Let
\[ \mathcal{Y} := \Phi(\mathcal{X}). \] (16)
Then \( \Phi \) is a map from \( \mathcal{X} \) to \( \mathcal{Y} \) and \( \mathcal{Y} \) is a Fréchet manifold. Let \( p = (u, P) \in \mathcal{X} \) be an arbitrary point. Let
\[ \varphi_p : \mathcal{X} \xrightarrow{\sim} T_p \mathcal{X} \] (17)
and
\[ \psi_{\Phi(p)} : \mathcal{X} \xrightarrow{\sim} T_{\Phi(p)} \mathcal{X} \] (18)
be canonical isomorphisms. Let
\[ \mathcal{U}_p := \varphi_p^{-1}(T_p \mathcal{X}) (= \mathcal{X}) \] (19)
and
\[ \mathcal{V}_{\Phi(p)} := \psi_{\Phi(p)}^{-1}(T_{\Phi(p)} \mathcal{Y}). \] (20)

Let \( H^k(\mathbb{R}^3/\mathbb{Z}^3) \) be the Sobolev space and
\[ W^{k_1, k_2}_n := C^{k_1}(I, H^{k_2}(\mathbb{R}^3/\mathbb{Z}^3)^n) \] (21)

Let \( k \in \mathbb{N} \). Let
\[ (W^{k,k+2}_3)_{\text{div}} := \{ v \mid v = \nabla \cdot w \ (\exists w \in W^{k,k+2}_3) \}. \] (22)

Introduce to \( \mathcal{Y} \) the relative topology as a subset of
\[ W^{k,k}_3 \times (W^{k,k+2}_3)_{\text{div}} \times H^{k+2}(\mathbb{R}^3/\mathbb{Z}^3)^3, \] (23)
which defines a seminorm \( p_k \) (it is in fact a norm) of \( \mathcal{Y} \). Introduce to \( T_{\Phi(p)} \mathcal{Y} \) the topology induced from \( (\psi_{\Phi(p)})^{-1}|_{T_{\Phi(p)} \mathcal{Y}} \), to \( T_p \mathcal{X} \) the topology induced from \( (\psi_{\Phi(p)})^{-1}|_{T_{\Phi(p)} \mathcal{Y}} \circ d\Phi(p) \), to \( \mathcal{U}_p \) the topology induced from \( (\psi_{\Phi(p)})^{-1}|_{T_{\Phi(p)} \mathcal{Y}} \circ d\Phi(p) \circ \varphi_p|_{\mathcal{U}_p} \), and to \( \mathcal{V}_{\Phi(p)} \) the topology induced from \( (\psi_{\Phi(p)})^{-1}|_{T_{\Phi(p)} \mathcal{Y}} \circ \psi_{\Phi(p)}|_{\mathcal{V}_{\Phi(p)}} \).

From above \( \mathcal{U}_p = \mathcal{X} \) and \( \mathcal{V}_{\Phi(p)} = \mathcal{X} \). The topologies are induced by a seminorm
\[ p_k \] such that \( p_k(r) \) implies \( r = 0 \) (for \( r \in \mathcal{Y} \)), \( q \mapsto d\Phi(q) \) is continuous with respect to the ordinary topology of \( \mathcal{U}_p \) and the topology of \( T_{\Phi(p)} \mathcal{V}_{\Phi(p)}(= T_{\Phi(p)} \mathcal{Y}) \) induced by \( p_k \).

Let \( Y \) be a Banach manifold and \( B \) a Banach space. Let \( L(q) : T_q Y \to B \) \((q \in Y)\) be a map. Then the set of accumulation points of the sequences

\[
\sum_i L(\gamma(\xi_i^j))\gamma(\xi_i^j)|E_i^j|,
\]

as \( j \to \infty \), where \( \gamma : [0, 1] \to Y \) runs over all smooth paths from \( p \in Y \) to \( q \in Y \), \( \{E_i^j\} \) all sequences of measurable sets of \([0, 1] \) such that \( \prod_i E_i^j = [0, 1] \) and \( \sup_i E_i^j \to 0 \) \((j \to \infty) \) and \( \xi_i^j \) all elements of \( E_i^j \), if exists, is denoted by \( \int_p L(q') \).

The set \( X \) equipped with the topology induced by \( p_k \) is, if exists, denoted by \( X^w \).

With the use of a result of [16] it is proved that there exist a sufficiently small convex neighbourhood \( \mathcal{X}_p \) of \( p \in \mathcal{U}_p \) with respect to the ordinary topology and a sufficiently small neighbourhood of \( 0 \in (T_{\Phi(p)} \mathcal{Y})^w \) identified with a sufficiently small neighbourhood \( \mathcal{Y}_{\Phi(p)}^w \) of \( \Phi(p) \in \mathcal{V}_{\Phi(p)}^w \) such that the multi-valued map \( q \in \mathcal{X}_p \mapsto \int_p d\Phi(q') \in \mathcal{Y}_{\Phi(p)}^w \) has a continuous branch. \( \mathcal{X}_p \) and \( \mathcal{Y}_p \) are identified with the corresponding neighbourhoods of \( 0 \in T_q \mathcal{X}_p \) \((q \in \mathcal{X}_p) \) and \( T_q T_q \mathcal{X}_p \) \((Q \in \mathcal{X}_p^2) \). The same argument holds for \( d(d\Phi) \). It follows that \( d\Phi : (\mathcal{X}_p^w)^w \to (T \mathcal{Y}_{\Phi(p)}^w)^w \) is a continuous map and the Fréchet derivative of \( \Phi : \mathcal{X}_p^w \to \mathcal{Y}_{\Phi(p)}^w \) is equal to \( d\Phi \) on \( (\mathcal{X}_p^2)^w \).

Similarly \( d(d\Phi) : (\mathcal{X}_p^w)^w \to (T T \mathcal{Y}_{\Phi(p)}^w)^w \) is a continuous map and the Fréchet derivative of \( d\Phi : (\mathcal{X}_p^w)^w \to (T \mathcal{Y}_{\Phi(p)}^w)^w \) is equal to \( d(d\Phi) \) on \( (\mathcal{X}_p^2)^w \). Let \( L((T_p \mathcal{X}_p)^w, (T_{\Phi(p)} \mathcal{Y}_{\Phi(p)}^w)^w) \) be the set of continuous linear operators from \( (T_p \mathcal{X}_p)^w \) to \( (T_{\Phi(p)} \mathcal{Y}_{\Phi(p)}^w)^w \). Then there exists \( M' > 0 \) such that

\[
||d\Phi(q') - d\Phi(q)||L((T_p \mathcal{X}_p)^w, (T_{\Phi(p)} \mathcal{Y}_{\Phi(p)}^w)^w) \leq M'||q' - q||\mathcal{X}_p^w,
\]

for \( q, q' \in \mathcal{X}_p^w \), and such that

\[
||\Phi(q') - \Phi(q) - d\Phi(q)(q' - q)||\mathcal{Y}_{\Phi(p)}^w \leq M'||q' - q||\mathcal{X}_p^w,
\]

for \( q, q' \in \mathcal{X}_p^w \).

Take the completions \( \mathcal{U}_p^k \) and \( \mathcal{V}_{\Phi(p)}^k \) of \( \mathcal{X}_p^w \) and \( \mathcal{Y}_{\Phi(p)}^w \) induced from \( p_k \). From above \( d\Phi \) extends to a continuous map, \( \Phi \) extends to a \( C^1 \)-map from \( \mathcal{U}_p^k \) to \( \mathcal{V}_{\Phi(p)}^k \) and the Fréchet derivative with respect to the topologies of \( \mathcal{U}_p^k \) and \( \mathcal{V}_{\Phi(p)}^k \) of the extended \( \Phi \) at \( q \) is equal to the value \( d\Phi(q) \) at \( q \) of the extended \( d\Phi \).
$d\Phi(p)$ extends to a topological isomorphism between the completions $T_p\mathcal{U}_p^k$ and $T_{\Phi(p)}\mathcal{V}_{\Phi(p)}^k$ of $(T_p\mathcal{X})^w$ and $(T_{\Phi(p)}\mathcal{Y})^w$ induced from $p_k$ so that since $\mathcal{X}_p$ is sufficiently small $d\Phi(q)$ ($q \in \mathcal{U}_p^k$) is a topological isomorphism. By Inverse Function Theorem there exist sufficiently small neighbourhoods $U_p$ and $V_{\Phi(p)}$ of $p \in \mathcal{U}_p^k$ and $\Phi(p) \in \mathcal{V}_{\Phi(p)}^k$ such that the extended

$$\Phi : U_p \xrightarrow{\simeq} V_{\Phi(p)}$$

is an isomorphism. Note that it is clear that $V_{\Phi(p)}$ ($p \in \mathcal{X}$) form a Banach manifold.

Replacing $I$ with an interval $J := [t_0, t_0 + T_0]$ (0 $\leq t_0 < T$ and $T_0 > 0$ is small) we define $\mathcal{X}^J$, $\mathcal{Y}^J$, $U_p^J$, $\Phi^J$ etc. in the same way as $\mathcal{X}$, $\mathcal{Y}$, $U_p$, $\Phi$ etc. Then it is shown that $U_p^J$ ($p \in \mathcal{X}^J$) form a Banach manifold.

We shall prove the local existence and uniqueness of a sufficiently smooth solution of equation (7) (of which smoothness depends on $k \in \mathbb{N}$).

Let $t_0 \in I$. Introduce to the inductive limits $\mathfrak{U}_{t_0}$ of $\bigcup_p U_p^J$ for $J \ni t_0$ and $\mathfrak{C}_{t_0}$ of $\bigcup J \mathfrak{Z}^{(k)J} := \{ (f,r_2,u_0) \in (W_3^{k,k})^J \times (W_3^{k,k+2})^J_{\text{div}} \times H^{k+2}(\mathbb{R}^3/\mathbb{Z}^3)^3 \}$ (28)

$$| \nabla \cdot u_0 = r_2(t_0) \}$$

(29)

for $J \ni t_0$ the natural topologies (the quotient topologies induced from the maps

$$\prod_p U_p^J \to \mathfrak{U}_{t_0}$$

and $\mathfrak{Z}^{(k)J} \to \mathfrak{C}_{t_0}$). Then the induced map $\tilde{\Phi}_{t_0} : \mathfrak{U}_{t_0} \to \mathfrak{C}_{t_0}$ is a homeomorphism. We obtain a bijection $\tilde{\Phi} : \mathcal{U} := \prod_{t_0} \mathfrak{U}_{t_0} \to \mathcal{C} := \prod_{t_0} \mathfrak{C}_{t_0}$ and introducing a sheaf structure to $\mathcal{C}$ induced from $\mathcal{U}$ through this bijection a sheaf isomorphism, where the topology of $I$ is generated by $J$'s. This proves the local existence and uniqueness of equation (7).

Let $\Omega \subset \mathbb{R}^3/\mathbb{Z}^3$. Let

$$W_{\Omega,n} := C^\infty(I, C^\infty(\Omega)^n)$$

(30)

$$(W_{\Omega,1})^{\nabla} := \{ v \mid v = \nabla w \ (\exists w \in W_{\Omega,1}) \}$$

(31)

$$(W_{\Omega,1})^{\nabla} := W_{\Omega,3} \times (W_{\Omega,1})^{\nabla}$$

(32)

Let $(f,r_2,u_0) \in \mathcal{X}$. Let $x \in \mathbb{R}^3/\mathbb{Z}^3$. Then there exist a compact neighbourhood $K_x$ of $x$ and $(u^{K_x}, \nabla p^{K_x}) \in \mathcal{X}_{K_x}$ satisfying equation (7) on $I \times K_x$. Consider a family $\{(u^{K_x}, \nabla p^{K_x})\}_x$. Since $\mathbb{R}^3/\mathbb{Z}^3$ is compact we may obtain a finite set $\{x_\lambda\}$ such that each $K_{x_\lambda}$ intersects with another in a set of Lebesgue measure 0 and $\bigcup \lambda \bigcup K_{x_\lambda} = \mathbb{R}^3/\mathbb{Z}^3$. It follows that there exists $(u, \nabla p) \in (L^2(I \times (\mathbb{R}^3/\mathbb{Z}^3)^3))^2$ that is smooth a.e. and satisfies equation (7) a.e. From above it is proved that
\( \Phi : \mathcal{X} \rightarrow \mathcal{Y} \) is a homeomorphism. Extend \( \Phi^{-1} \) to a continuous map from \( \mathcal{Y} (\subset \mathcal{Y}) \) to

\[
\{(u, \nabla p) \in (L^2(I \times (\mathbb{R}^3/\mathbb{Z}^3))^2 \mid u, \nabla p \text{ are smooth a.e.}\}.
\]

Then considering the convolutions with mollifiers it is proved that \( (u, \nabla p) \in \Phi^{-1}(\mathcal{Y}) \) and thus \( (u, \nabla p) = \Phi^{-1}((f, r_2, u_0)) \). Let \( (s, y) \in I \times (\mathbb{R}^3/\mathbb{Z}^3) \) be a singular point of \((u, \nabla p)\). Take another finite decomposition \( K_y \cup (\bigcup_{\mu} K_{x_{\mu}}) = \mathbb{R}^3/\mathbb{Z}^3 \) and construct \((u_1, \nabla p_1)\) that is smooth at \((s, y)\) and satisfies equation (7) a.e. Since \((u, \nabla p) = \Phi^{-1}((f, r_2, u_0)) = (u_1, \nabla p_1)\) it follows that \((u, \nabla p)\) is smooth at \((s, y)\). This contradiction shows that \((u, \nabla p) \in \mathcal{X} \) and \((f, r_2, u_0) = \Phi((u, \nabla p)) \in \mathcal{Y} \). Hence \( \mathcal{X} = \mathcal{Y} \).

Since \((f, 0, u_0) \in \mathcal{X} = \mathcal{Y} \) there exists \( p \in \mathcal{X} \) such that \( \Phi(p) = (f, 0, u_0) \), which defines a smooth solution of (7). Further since \( \Phi : \mathcal{X} \rightarrow \mathcal{Y} \) is a homeomorphism the solution is unique. The assertion of Theorem 2 follows.

We note that \( \{X_N\}_N \) such that \( \bigcup_N X_N = \mathcal{X} \) is not unique. In our case \( \{X_N\}_N \)

is taken to be \( \{\mathcal{X}_p\}_p \in \mathcal{X} \) and the existence and uniqueness of local solutions are obtained. In summary we used the following Navier-Stokes conditions (which are not assumptions) and conclude \( \Phi \) is a homeomorphism.

1. \( \Phi : \mathcal{X} \rightarrow \mathcal{Y} \) is a \( C^\infty \)-map such that \( d\Phi(p) : T_p \mathcal{X} \rightarrow T_{\Phi(p)} \mathcal{Y} \) is a linear isomorphism.

2. Each seminorm \( p_k \) of \( \mathcal{Y} \), which is induced from the relative topology from

\[
W_3^{k,k} \times (W_3^{k,k+2})_{\text{div}} \times H^{k+2}(\mathbb{R}^3/\mathbb{Z}^3)^3,
\]

is actually a norm, that is, it satisfies the condition that \( p_k(r) = 0 \) implies \( r = 0 \).

3. The above holds if we replace \( I \) with \( J \).

4. For any \( r \in \mathcal{X} \) there exists \( q \in \bigcup_p U_p^J \) such that \( \Phi'(q)|_t = r(t) \ (t_0 \leq t < t_0 + t_1) \) for small \( t_1 \).

5. Let \((f, r_2, u_0) \in \mathcal{X} \). Let \( x \in \mathbb{R}^3/\mathbb{Z}^3 \). Then there exist a compact neighbourhood \( K_x \) of \( x \) and \((u^K_x, \nabla p^K_x) \in \mathcal{X}_{K_x} \) satisfying equation (7) on \( I \times K_x \).

From Theorem 2 it is easy to prove the following corollary.
Corollary 3. Let $\nu > 0$. Let $I := [0, \infty)$. Let $u_0 \in C^\infty(\mathbb{R}^3 / \mathbb{Z}^3)^3$ such that
$\nabla \cdot u_0 = 0$. Then there exists a unique $(u, \nabla p) \in W_3 \times (W_1)_\nu$ such that
\[
\frac{\partial u}{\partial t} = -(u \cdot \nabla)u + \nu \Delta u - \nabla p,
\]
\[
\nabla \cdot u = 0,
\]
\[
u|_{t=0} = u_0.
\] (35)

2 Preliminaries

Definition 4. Let $Z$ be a Banach space. Let $T_s \ (s \geq 0)$ be a bounded linear operator on $Z$. A $C^0$-semigroup is a family $\{T_s\}$ such that
\[
T_{s_1}T_{s_2} = T_{s_1+s_2} \ (s_1, s_2 \geq 0),
\]
\[
T_0 = \text{Id},
\]
and such that
\[
\lim_{s \to s_0} T_s x = T_{s_0} x \tag{38}
\]
for each $s_0 \geq 0$ and each $x \in Z$.

Definition 5. Let $Z$ be a Banach space. Let $\{T_s\}$ be a $C^0$-semigroup on $Z$. The infinitesimal generator $A$ of $\{T_s\}$ is a linear operator of which domain is the set $D(A) := \{z \in Z \mid \lim_{s \to +0} \frac{1}{s}(T_s - I)z \text{ exists in } Z\}$ such that for $z \in D(A)$
\[
Az := \lim_{s \to +0} \frac{1}{s}(T_s - I)z. \tag{39}
\]

Although it is not used we refer to the following Hille-Yosida theorem [8], Section 12.3, Theorem 12.3.2, of which expression is slightly changed.

Theorem 6. Let $Z$ be a Banach space. Let $A$ be a closed linear operator on $Z$. Then $A$ generates a $C^0$-semigroup if and only if the domain of $A$ is dense and there exist $M > 0$ and $\beta \in \mathbb{R}$ such that
\[
||(\lambda - A)^{-k}|| \leq M(\lambda - \beta)^{-k} \ (\lambda > \beta), \tag{40}
\]
for $k = 1, 2, \ldots$.

The following definition, of which expression is slightly changed, is given in [10].

Definition 7 (see [10]). Let $Z$ be a Banach space. Let $T > 0$. Assume a linear operator $A(t)$ on $Z$ generates a $C^0$-semigroup for each $t \in [0, T]$. The family $\{A(t)\}$ is stable if there exist $M > 0$ and $\beta \in \mathbb{R}$ such that
\[
\left| \prod_{j=1}^{k} (\lambda - A(t_j))^{-1} \right| \leq M(\lambda - \beta)^{-k} \ (\lambda > \beta), \tag{41}
\]
for any finite family \( \{ t_j \} \) with \( 0 \leq t_1 \leq \cdots \leq t_k \leq T \) (\( k = 1, 2, \ldots \)). The product \( \prod \) is time-ordered, i.e. a factor with larger \( t_j \) stands to the left of ones with smaller \( t_j \).

The following, of which expression is slightly changed, is given in [10], Section 3, Proposition 3.3, formula (3.2).

**Lemma 8.** Let \( Z \) be a Banach space. Let \( A(t) \) be a linear operator on \( Z \) which generates a \( C^0 \)-semigroup for each \( t \in [0, T] \). If \( \{ A(t) \} \) is stable with \( M > 0 \) and \( \beta \in \mathbb{R} \) then

\[
|| \prod_{j=1}^{k} e^{s_j A(t_j)} || \leq Me^{\beta \sum_{j=1}^{k} s_j},
\]

for \( 0 \leq t_1 \leq \cdots \leq t_k \leq T \) and \( s_j \geq 0 \). The product \( \prod \) is time-ordered, i.e. a factor with larger \( t_j \) stands to the left of ones with smaller \( t_j \).

The following is a special case of a result of [10].

**Theorem 9 (T. Kato).** Let \( T > 0 \). Let \( Z \) be a Banach space. Let \( A(t) \) be a linear operator on \( Z \). Let the domain of \( A(t) \) be equal to \( Z \). Assume \( A(t) \) generates a \( C^0 \)-semigroup for each \( t \in [0, T] \) and that \( \{ A(t) \} \) is stable. Let \( f \in C^0([0, T], Z) \) and \( u_0 \in Z \) then the equation

\[
\begin{cases}
\frac{du}{dt} = A(t)u + f, \\
u|_{t_0} = u_0,
\end{cases}
\]

has a solution \( u \in C^1([0, T], Z) \).

Let \( B(Z) \) be the set of bounded operators on \( Z \).

**Corollary 10.** Let \( Z \) be a Banach space. Let \( t_0 < T_0 \) (resp. \( T_0 < t_0 \)). Assume \( A(t) \in B(Z) \) for each \( t \in [t_0, T_0] \) (resp. for each \( t \in [T_0, t_0] \)). Let \( f \in C^0([t_0, T_0], Z) \) (resp. \( f \in C^0([T_0, t_0], Z) \)) and \( u_0 \in Z \). Assume there exists \( C > 0 \) such that \( ||A(t)||_{B(Z)} \leq C \) for any \( t \in [t_0, T_0] \) (resp. for any \( t \in [T_0, t_0] \)). Then the equation

\[
\begin{cases}
\frac{du}{dt} = A(t)u + f, \\
u|_{t_0} = u_0,
\end{cases}
\]

has a solution \( u \in C^1([t_0, T_0], Z) \) (resp. \( u \in C^1([T_0, t_0], Z) \)).

The following is a special case of a result of [11].

**Theorem 11 (K. Deimling).** Let \( Z \) be a Banach space. Let \( t_0 < T_0 \) (resp. \( T_0 < t_0 \)). Assume \( A(t) \in B(Z) \) for each \( t \in [t_0, T_0] \) (resp. for each \( t \in [T_0, t_0] \)).
Let \( f \in C^0([t_0, T_0], Z) \) (resp. \( f \in C^0([T_0, t_0], Z) \)) and \( u_0 \in Z \). Assume there exists \( C > 0 \) such that \( \|A(t)\|_B(Z) \leq C \) for any \( t \in [t_0, T_0] \) (resp. for any \( t \in [T_0, t_0] \)). Then the equation

\[
\begin{aligned}
\frac{du}{dt} &= A(t)u + f, \\
|u|_{t_0} &= u_0,
\end{aligned}
\]

has at most one solution \( u \in C^1([t_0, T_0], Z) \) (resp. \( u \in C^1([T_0, t_0], Z) \)).

The following theorem, of which expression is slightly changed, is given in [12], Chapter I, section 5, Theorem 5.2.

**Theorem 12.** Let \( X, Y \) be Banach spaces, \( O \) an open subset of \( X \) and let \( \xi : O \to Y \) a \( C^p \)-morphism with \( p \in \mathbb{N} \). Assume that for a point \( a_0 \in O \) the derivative \( d\xi(a_0) : X \to Y \) is a topological linear isomorphism. Then \( \xi \) is a local \( C^p \)-isomorphism at \( a_0 \).

# 3 Nonlinear ordinary differential equations (alternate proof)

We give an alternate proof of the following well-known theorem (see e.g. [2]).

**Theorem 13.** Let \( X \) be a Banach space. Let \( f : \mathbb{R} \times X \ni (t, u_1) \to f(t, u_1) \in X \) be \( C^0 \) in \( t \) and \( C^1 \) in \( u_1 \). Let \( u \in X \) be arbitrary. Let the Fréchet derivative \( D_x f(t, u) \) of \( f(t, \cdot) \) at \( u \) be \( C^0 \) in \( t \). Then the equation

\[
\begin{aligned}
\frac{du}{dt} &= f(t, u), \\
|u|_{t=0} &= u_0,
\end{aligned}
\]

has an unique solution \( u \in C^1(I_1, X) \) for some \( I_1 = (a, b) \) \((a < 0 < b)\) that is maximal in this property.

Let \( J \) a compact interval in \( \mathbb{R} \) containing \( t_0 \in \mathbb{R} \). Let \( u \in C^1(J, X) \). We prove the following lemma.

**Lemma 14.** The map

\[
\begin{aligned}
h &\mapsto \left[ \frac{dh}{dt} - \frac{D_x f(t, u)h}{|h|_{t=t_0}} \right]
\end{aligned}
\]

is a topological linear isomorphism from \( C^1(J, X) \) to \( C^0(J, X) \times X \).

**Proof.** Consider the equation

\[
\begin{aligned}
\frac{dh}{dt} &= D_x f(t, u)h + r, \\
h|_{t=t_0} &= h_0,
\end{aligned}
\]
where \( r \in C^0(J, X) \), \( h_0 \in X \). Let \( B(X) \) be the set of bounded operators on \( X \). Then by definition \( D_x f(t, u) \in B(X) \) for each \( t \in J \) and there exists \( C > 0 \) such that \( \| D_x f(t, u) \|_{B(X)} \leq C \) for any \( t \in J \). Thus by Corollary 10 and Theorem 11 equation (48) has an unique solution in \( C^1(J, X) \). The map

\[
    h \mapsto \left[ \frac{dh}{dt} - D_x f(t, u)h \right]_{h|^t=t_0}
\]

(49)

is a continuous linear isomorphism from \( C^1(J, X) \) to \( C^0(J, X) \times X \) and by the open mapping principle a topological linear isomorphism. The assertion follows. \( \square \)

Let \( \mathcal{X}^J := C^1(J, X) \), \( \mathcal{Y}^J := C^0(J, X) \times X \). Let \( p := u \in \mathcal{X}^J \). Define

\[
    \Phi^J : \mathcal{X}^J \ni u_1 \mapsto (F^J(u_1), G^J(u_1)) \in \mathcal{Y}^J,
\]

(50)

where

\[
    \begin{cases}
        F^J(u_1) := \frac{du_1}{dt} - f(t, u_1), \\
        G^J(u_1) := u_1|_t=t_0.
    \end{cases}
\]

(51)

\( F^J, G^J \) are \( C^1 \) in \( u_1 \). At \( p = u \in \mathcal{X}^J \), by calculation

\[
    d\Phi^J(p) : h \mapsto \left[ \frac{dh}{dt} - D_x f(t, u)h \right]_{h|^t=t_0}.
\]

(52)

By Lemma 14 \( d\Phi^J(p)(h) \) \((p \in \mathcal{X}^J)\) is a topological isomorphism from the tangent space \( T_p \mathcal{X}^J \) of \( \mathcal{X}^J \) at \( p \) to the tangent space \( T_{\Phi^J(p)} \mathcal{Y}^J \) of \( \mathcal{Y}^J \) at \( \Phi^J(p) \). Thus by Theorem 12 \( \Phi^J \) is a local diffeomorphism \((X = O = \mathcal{X}^J, Y = \mathcal{Y}^J, \xi = \Phi^J, a_0 = p)\).

We prove the local existence and uniqueness of a solution of equation (46). Let \( \mathcal{U}_{t_0}, \mathcal{C}_{t_0} \) be the inductive limits of \( \mathcal{X}^J \), \( \mathcal{Y}^J \) for \( J \ni t_0 \). Introduce to them the natural topologies (the quotient topologies induced from \( \prod_J \mathcal{X}^J \to \mathcal{U}_{t_0} \) and \( \prod_J \mathcal{Y}^J \to \mathcal{C}_{t_0} \)) so that \( \{\Phi^J\}_J \) induces a local diffeomorphism \( \tilde{\Phi}_{t_0} \). \( \mathcal{U}_{t_0} \) is connected and \( \mathcal{C}_{t_0} \) is simply connected.

**Lemma 15.** \( \tilde{\Phi}_{t_0} \) is surjective (and thus \( \tilde{\Phi}_{t_0} \) is a homeomorphism).

**Proof.** Let \((r, u_0) \in \mathcal{Y}^J \). From the definition

\[
    \mathcal{X}^J \ni u_1 \mapsto (F^J(u_1)|_{t_0}, G^J(u_1)) \in X \times X
\]

(53)

is surjective. Thus there exists \( u' \in \mathcal{X}^J \) such that \((F^J(u')|_{t_0}, G^J(u')) = (r|_{t_0}, u_0)\). Since \( \Phi^J \) is an open map there exists \( \epsilon > 0 \) such that any \( y \in \mathcal{Y}^J \) with

\[
    \| y - (F^J(u'), G^J(u')) \|_{C^0(J, X) \times X} < \epsilon
\]

(54)
is in $\text{Im} \Phi^J$. On the other hand for $s \in \mathbb{R}$ with $|s|$ small

$$|| (r|_{t=t_0+s}, u_0) - (F^J(u')|_{t=t_0+s}, G^J(u')) ||_{X \times X} < \epsilon. \quad (55)$$

Thus there exists $u \in \mathcal{X}^J$ such that $(F^J(u)|_t, G^J(u)) = (r|_t, u_0)$ on a neighbourhood of $t_0$. The image of $\Phi_{t_0}$ contains the equivalence class of $(r, u_0)$ at $t_0$. Hence $\Phi_{t_0}$ is surjective. The assertion follows. \qed

**Remark 16.** The above results are also proved by successive approximation method.

**Proof of Theorem 13.** Now we obtain a bijection $\tilde{\Phi} : \mathcal{U} = \bigsqcup_{t_0} \mathcal{U}_{t_0} \to \mathcal{C} = \bigsqcup_{t_0} \mathcal{C}_{t_0}$ and introducing a sheaf structure to $\mathcal{C}$ (induced from $\mathcal{U}$) through this bijection a sheaf isomorphism. Thus for $u_0 \in X$ there exists a section $\tilde{p}$ defined on a maximal $I_1 := (a, b)$ such that

$$\tilde{\Phi}(\tilde{p}) = \left( \frac{dp}{dt} - f(t, \tilde{p}), \tilde{p}|_{t=0} \right) = (0, u_0), \quad (56)$$

where $\tilde{p}$ is locally unique because $\tilde{\Phi}$ is an isomorphism. Hence $\tilde{p}$ is an unique solution of equation (46) defined on an interval $I_1$ that is maximal in this property. The assertion follows. \qed

## 4 Hodge Theory on $\mathbb{R}^3/\mathbb{Z}^3$

Let $\nabla \cdot (\cdot) : u \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3 \mapsto \nabla \cdot u \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3)$. Let

$$(C^\infty(\mathbb{R}^3/\mathbb{Z}^3))_\nabla := \{ v \mid v = \nabla w \ (\exists w \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3)) \}. \quad (57)$$

The following is a special case of [3], Chapter VI, Section 3.3, (3.16) Theorem.

**Theorem 17.** $C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3 = \text{Ker}(\nabla \cdot (\cdot)) \oplus (C^\infty(\mathbb{R}^3/\mathbb{Z}^3))_\nabla$.

## 5 Linear evolution equations on $\mathbb{R}^3/\mathbb{Z}^3$

The following theorem, of which expression is slightly changed, is also known as Hille-Yosida theorem (see [20], Chapter IX, Section 8).

**Theorem 18.** Let $X$ be a Banach space and $\tilde{A}$ a linear operator on $X$. Assume the domain of $\tilde{A}$ is dense in $X$. Then $\tilde{A}$ generates a contraction $C^0$-semigroup if and only if $\tilde{A}$ is dissipative with respect to a semiscalar product and the range of $\text{Id} - \tilde{A}$ is equal to $X$.

Let $\mathcal{P} : C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3 \to \text{Ker}(\nabla \cdot (\cdot))$ be the projection. For $n \in \mathbb{N}$ let

$$W_n := C^\infty(I, C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^n). \quad (58)$$
We shall prove the existence and uniqueness of a solution of the equation

\[
\begin{cases}
  \dot{h} + \mathcal{P}((u \cdot \nabla)\mathcal{P}h + (\mathcal{P}h \cdot \nabla)u - \nu \Delta \mathcal{P}h) - g = 0, \\
  h(0) \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3,
\end{cases}
\]

for \( g \in W_3 \).

Let \( k \in \mathbb{Z}_{\geq 0} \). Let \( I := [0, T] \) and \( u \in W_3 \). Let \( H^k(\mathbb{R}^3/\mathbb{Z}^3)^3 \) be the Sobolev space. Define a linear operator \( A'_k(t) \) on \( H^k(\mathbb{R}^3/\mathbb{Z}^3)^3 \) by

\[
-\dot{A}'_k(t)h := \mathcal{P}((u \cdot \nabla)\mathcal{P}h + (\mathcal{P}h \cdot \nabla)u - \nu \Delta \mathcal{P}h)
\]

for \( h \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3 \). Since the adjoint \((A'_k(t))^*\) of \( A'_k(t) \) is densely defined \( A'_k(t) \) is closable. Let \( A_k(t) \) be the closure of \( A'_k(t) \). For \( k_1, k_2 \in \mathbb{Z}_{\geq 0} \cup \{\infty\} \) let

\[
W^k_{n_1, n_2} := C^{k_1}(I, H^{k_2}(\mathbb{R}^3/\mathbb{Z}^3)^n).
\]

**Lemma 19.** Let \( I := [0, T] \) \((T > 0)\). Let \( \nu > 0 \). Assume \( u \in W_3 \). Then the equation

\[
\dot{h} - A_k(t)h = 0
\]

has at most one solution \( h \in W^1_{k, k} \) for any initial condition \( h(0) \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3 \).

**Proof.** Let \( h \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3 \). Observe that \( A_k(t)h = \mathcal{P}(-\nu(\nabla - A(t))^2)\mathcal{P}h + \mathcal{P}\mathcal{B}(t)\mathcal{P}h \) for some \( W_1 \)-coefficiential \( 3 \times 3 \) and \( 3 \times 3 \) matrices \( A(t), \mathcal{B}(t) \). Thus by an elementary argument

\[
\text{Re} \langle -A_k(t)h, h \rangle_{H^k(\mathbb{R}^3/\mathbb{Z}^3)^3} \geq -c < h, h \rangle_{H^k(\mathbb{R}^3/\mathbb{Z}^3)^3}
\]

for some \( c > 0 \). Taking limit formula (63) also holds for \( h \in D(\mathcal{A}_k(t)) \), where \( D(\mathcal{A}_k(t)) \) is the domain of \( -A_k(t) \).

Let \( h(t) \) be the solution of equation (62) and \( H = e^{-ct}h \). Observe that \( h \) depends on \( t \). Then

\[
\langle \dot{H}, H \rangle_{H^k(\mathbb{R}^3/\mathbb{Z}^3)^3} = \langle (A_k(t) - c)H, H \rangle_{H^k(\mathbb{R}^3/\mathbb{Z}^3)^3}.
\]

Assume \( h(0) = 0 \). Let \( t_0 \in [0, T] \). By formula (63)

\[
||H(t_0)||_{H^k(\mathbb{R}^3/\mathbb{Z}^3)^3}^2 - ||H(0)||_{H^k(\mathbb{R}^3/\mathbb{Z}^3)^3}^2 = 2 \int_0^{t_0} \text{Re} \langle (A_k(t) - c)H, H \rangle_{H^k(\mathbb{R}^3/\mathbb{Z}^3)^3} dt
\]

\[
\leq 0.
\]

It follows that

\[
||H(t_0)||_{H^k(\mathbb{R}^3/\mathbb{Z}^3)^3}^2 \leq ||H(0)||_{H^k(\mathbb{R}^3/\mathbb{Z}^3)^3}^2 = 0.
\]
Hence \( h(t_0) = 0 \). Since \( t_0 \in [0, T] \) is arbitrary \( h = 0 \). Assume \( h_1, h_2 \) are two solutions. Then \( h_2(0) - h_1(0) = 0 \) and \( h_2 - h_1 \) is a solution of the equation (62). Thus the above argument shows that \( h_2 - h_1 = 0 \). From this the assertion follows. \( \square \)

**Lemma 20.** \( A_k(t) \) generates a \( C^0 \)-semigroup for each \( t \in I \).

**Proof.** Take \( c \) as in Lemma 19. Since by definition the domain of \( A_k(t) \) is dense, so is that of \( (A_k(t) - c) \). Observe that

\[
\text{Re} \langle (A_k(t) - c)h, h \rangle_{H^k(\mathbb{R}^3/\mathbb{Z}^3)^3} \leq 0 \tag{69}
\]

for all \( h \in D(A_k(t) - c) \), where \( D(A_k(t) - c) \) is the domain of \( (A_k(t) - c) \), so \((A_k(t) - c)\) is dissipative. In particular the image of \( \text{Id} - (A_k(t) - c) \) is closed. The adjoint operator \((\text{Id} - (A_k(t) - c))^* \) of \( \text{Id} - (A_k(t) - c) \) is clearly injective. Hence the image of \( \text{Id} - (A_k(t) - c) \) is the whole space \( H^k(\mathbb{R}^3/\mathbb{Z}^3)^3 \). By Theorem 18, \( (A_k(t) - c) \) generates a contraction \( C^0 \)-semigroup. The assertion follows. \( \square \)

**Lemma 21.** Let \( k, k' \in \mathbb{Z}_{\geq 0} \). Let \( \{e^{sA_k(t)}\}_{s \geq 0} \) be a \( C^0 \)-semigroup on \( H^k(\mathbb{R}^3/\mathbb{Z}^3)^3 \) generated by \( A_k(t) \) for each \( t \). Then \( e^{sA_k(t)}h_0 = e^{sA_{k'}(t)}h_0 \) for any \( h_0 \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3 \) and \( s \in I := [0, T] \).

**Proof.** Assume without loss of generality \( k \leq k' \). Observe that by assumption \( h_0 \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3 \) so that \( h_1(s, x) := e^{sA_k(t)}h_0 \in W^{1,k}_3 \) and \( h_2(s, x) := e^{sA_{k'}(t)}h_0 \in W^{1,k'}_3 \). Then since \( -A_k(t) \big|_{H^{k'}(\mathbb{R}^3/\mathbb{Z}^3)^3} = -A_{k'}(t) \) it follows that \( h_1 \) and \( h_2 \) are solutions of

\[
\begin{cases}
\frac{\partial h}{\partial t} + A_k(t)h = 0, \\
h(0) = h_0.
\end{cases}
\tag{70}
\]

Since \( A_k(t) \) is dissipative an elementary argument shows that the solution \( h \in W^{1,k}_3 \) of this equation is unique. Thus \( e^{sA_k(t)}h_0 = h_1(s, x) = h_2(s, x) = e^{sA_{k'}(t)}h_0 \) for \( s \in I \). The assertion follows. \( \square \)

Let \( B(H^k(\mathbb{R}^3/\mathbb{Z}^3)^3) \) be the set of continuous linear operators on \( H^k(\mathbb{R}^3/\mathbb{Z}^3)^3 \).

**Theorem 22.** Assume

\[
\| \prod_{j=1}^J (\lambda - A_k(t_j))^{-1} \|_{B(H^k(\mathbb{R}^3/\mathbb{Z}^3)^3)} \leq M_k(\lambda - \beta_k)^{-1} \tag{71}
\]

for \( 0 \leq t_1 \leq \cdots \leq t_J \leq T \) \((J = 1, 2, \ldots)\), where the product \( \prod \) is time-ordered, i.e. a factor with larger \( t_j \) stands to the left of ones with smaller \( t_j \). Then there exists an operator-valued function \( U(t, s) \) \((0 \leq s \leq t \leq T)\) on \( C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3 \) that satisfies the following.

(a) \((s, t) \mapsto U(t, s)h_0 \) \((h_0 \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3) \) is continuous, \( U(s, s) = \text{Id} \),

\[
\|U(t, s)\|_{B(H^k(\mathbb{R}^3/\mathbb{Z}^3)^3)} \leq M_ke^{\beta_k(t-s)}. \tag{72}
\]
(b) If \( s \leq r \leq t \) then \( U(t, s) = U(t, r)U(r, s) \).
(c) For \( h_0 \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3 \) and \( s \in [0, T) \),
\[
D_t^+ U(t, s)h_0|_{t=s} = A_0(s)h_0,
\]
where \( D^+ \) denotes the right derivative.
(d) For \( h_0 \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3 \) and \( 0 \leq s \leq t \leq T \),
\[
\frac{\partial}{\partial s} U(t, s)h_0 = -U(t, s)A_0(s)h_0.
\]
(e) For \( h_0 \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3 \) and \( 0 \leq s \leq t \leq T \),
\[
\frac{\partial}{\partial t} U(t, s)h_0 = A_0(t)U(t, s)h_0.
\]

Proof. By Lemma 20, \( A_k(t) \) generates a \( C^0 \)-semigroup \( \{e^{sA_k(t)}\}_{s \geq 0} \) and since \( C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3 \) is dense in \( H^k(\mathbb{R}^3/\mathbb{Z}^3)^3 \), it is obtained by Lemma 21 that
\[
e^{sA_0(t)}|_{H^k(\mathbb{R}^3/\mathbb{Z}^3)^3} = e^{sA_k(t)}.
\]
By Lemma 8,
\[
\left\| \prod_{j=1}^{J} e^{s_jA_0(t_j)} \right\|_{H^k(\mathbb{R}^3/\mathbb{Z}^3)^3} \leq B(\mathbb{H}^k(\mathbb{R}^3/\mathbb{Z}^3)^3)
\]
\[
\left\| \prod_{j=1}^{J} e^{s_jA_k(t_j)} \right\|_{B(\mathbb{H}^k(\mathbb{R}^3/\mathbb{Z}^3)^3)}
\]
\[
\leq M_k e^{\beta_k \sum_{j=1}^{J} s_j}\]
for \( 0 \leq t_1 \leq \cdots \leq t_J \leq T \) and \( s_j \geq 0 \). The product \( \prod \) is time-ordered, i.e. a factor with larger \( t_j \) stands to the left of ones with smaller \( t_j \).

Observe that \( A_0(t)|_{H^k(\mathbb{R}^3/\mathbb{Z}^3)^3} = A_k(t) \). Let \( L(H^{k+2}(\mathbb{R}^3/\mathbb{Z}^3)^3, H^k(\mathbb{R}^3/\mathbb{Z}^3)^3) \) be the set of continuous linear operators from \( H^{k+2}(\mathbb{R}^3/\mathbb{Z}^3)^3 \) to \( H^k(\mathbb{R}^3/\mathbb{Z}^3)^3 \). Let \( A_{0,n}(t) = A_0(T|nt/T|/n) \), where \( [\cdot] \) denotes the Gauss symbol. Then
\[
\left\| A_{0,n}(t) - A_0(t) \right\|_{L(H^{k+2}(\mathbb{R}^3/\mathbb{Z}^3)^3, H^k(\mathbb{R}^3/\mathbb{Z}^3)^3)} \to 0 \ (n \to \infty).
\]
Let
\[
U_n(t, s) = e^{(t-s)A_0(k'T/n)}, \quad (k'T/n \leq s \leq t \leq (k' + 1)T/n),
\]
\[
U_n(t, s) = e^{(t-l't/n)A_0(l'T/n)}e^{(l'T/n)A_0((l'-1)T/n)} \cdots e^{(T/n)A_0((k'+1)T/n)}e^{((k'+1)T/n-s)A_0(k'T/n)},
\]
(\( k'T/n \leq s < (k' + 1)T/n, l'T/n \leq t < (l' + 1)T/n, k' < l' \)).
By inequality (77)-(79),
\[ \|U_n(t, s)\|_{B(H^k(\mathbb{R}^3/\mathbb{Z}^3)^3)} \leq M_k e^{\beta_k(t-s)}. \]
(85)
Thus \( U_n(t, s) \) satisfies (a), (b). For \( h_0 \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3 \) and \( t \neq k''T/n \ (k'' = 0, 1, \ldots, n) \),
\[ \frac{\partial}{\partial r} U_n(t, s)h_0 = A_{0,n}(t)U_n(t, s)h_0, \]
(86)
and for \( h_0 \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3 \) and \( s \neq l''T/n \ (l'' = 0, 1, \ldots, n) \),
\[ \frac{\partial}{\partial s} U_n(t, s)h_0 = -U_n(t, s)A_{0,n}(s)h_0. \]
(87)
Let \( h_0 \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3 \). Then by inequality (85),
\[ \|U_n(t, s)h_0 - U_m(t, s)h_0\|_{H^k(\mathbb{R}^3/\mathbb{Z}^3)^3} \]
(88)
\[ = \| - \int_s^t \frac{\partial}{\partial r} U_n(t, r)U_m(r, s)h_0 dr\|_{H^k(\mathbb{R}^3/\mathbb{Z}^3)^3} \]
(89)
\[ = \| \int_s^t U_n(t, r)(A_{0,n}(r) - A_{0,m}(r))U_m(r, s)h_0 dr\|_{H^k(\mathbb{R}^3/\mathbb{Z}^3)^3} \]
(90)
\[ \leq M_kM_{k+2}e^{\max\{\beta_k, \beta_{k+2}\}(t-s)}\|h_0\|_{H^{k+2}(\mathbb{R}^3/\mathbb{Z}^3)^3} \]
\[ \times \int_s^t \|A_{0,n}(r) - A_{0,m}(r)\|_{L^1(H^{k+2}(\mathbb{R}^3/\mathbb{Z}^3)^3, H^k(\mathbb{R}^3/\mathbb{Z}^3)^3)} dr \]
(91)
\[ \rightarrow 0 \ (n, m \rightarrow \infty). \]
(92)
\( k \in \mathbb{Z}_{\geq 0} \) is arbitrary and thus for any \( h_0 \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3 \) and \( 0 \leq s \leq t \leq T \),
\[ U(t, s)h_0 := \lim_{n \rightarrow \infty} U_n(t, s)h_0 \]
(93)
exists. Note that \((s, t) \mapsto U(t, s)\) is continuous because the convergence is uniform in \( 0 \leq s \leq t \leq T \). Since \( U_n(t, s) \) satisfies (a), (b) so does \( U(t, s) \). For \( h_0 \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3 \),
\[ ||U_n(t, s)h_0 - e^{(t-s)A_0(s)}h_0||_{H^k(\mathbb{R}^3/\mathbb{Z}^3)^3} \]
(94)
\[ = || - \int_s^t \frac{\partial}{\partial r} U_n(t, r)e^{(r-s)A_0(s)}h_0 dr||_{H^k(\mathbb{R}^3/\mathbb{Z}^3)^3} \]
(95)
\[ = || \int_s^t U_n(t, r)(A_{0,n}(r) - A_0(r))e^{(r-s)A_0(s)}h_0 dr||_{H^k(\mathbb{R}^3/\mathbb{Z}^3)^3} \]
(96)
\[ \leq M_kM_{k+2}e^{\max\{\beta_k, \beta_{k+2}\}(t-s)}||h_0||_{H^{k+2}(\mathbb{R}^3/\mathbb{Z}^3)^3} \]
(97)
\[ \times \int_s^t \|A_{0,n}(r) - A_0(r)\|_{L^1(H^{k+2}(\mathbb{R}^3/\mathbb{Z}^3)^3, H^k(\mathbb{R}^3/\mathbb{Z}^3)^3)} dr. \]
(98)
It follows that for $h_0 \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3$,
\[
\|U(t, s)h_0 - e^{(t-s)A_0(s)}h_0\|_{H^k(\mathbb{R}^3/\mathbb{Z}^3)^3} \\
\leq M_k M_{k+2} e^{\max\{\beta_k, \beta_{k+2}\}(t-s)}\|h_0\|_{H^{k+2}(\mathbb{R}^3/\mathbb{Z}^3)^3} \\
\times \int_s^t \||A_0, n(r) - A_0(r)||_{L(H^{k+2}(\mathbb{R}^3/\mathbb{Z}^3)^3, H^k(\mathbb{R}^3/\mathbb{Z}^3)^3)} dr.
\] (100) (101) (102)

Since $k \in \mathbb{Z}_{\geq 0}$ is arbitrary this proves (c). Similarly it is shown that for $h_0 \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3$,
\[
D^- U(t, s)h_0|_{s=t} = -A_0(t)h_0,
\] (103)
where $D^-$ denotes the left derivative. For $h_0 \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3$, $s < t$ it is obtained by (c) and the continuity of $U(t, s)$ that
\[
\frac{1}{\epsilon} (U(t, s + \epsilon)h_0 - U(t, s)h_0) \\
= U(t, s + \epsilon)\frac{1}{\epsilon}(h_0 - U(s + \epsilon, s)h_0) \\
\to -U(t, s)A_0(s)h_0 \quad (\epsilon \to +0).
\] (104) (105) (106)

For $h_0 \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3$, $s \leq t$ it is obtained by formula (103) that
\[
\frac{1}{\epsilon} (U(t, s)h_0 - U(t, s - \epsilon)h_0) \\
= U(t, s)\frac{1}{\epsilon}(h_0 - U(s, s - \epsilon)h_0) \\
\to -U(t, s)A_0(s)h_0 \quad (\epsilon \to +0).
\] (107) (108) (109)

They prove (d). (e) follows from (c) and (d).

\[\square\]

**Lemma 23.** Let $I := [0, T]$ ($T > 0$). Let $\nu > 0$. Assume $u, g \in W_3$. Then the equation
\[
\dot{h} - A_0(t)h - g = 0
\] (110)
has a solution $h \in W^1_{3, \infty}$ that is uniquely determined from $h(0) \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3$.

**Proof.** By Theorem 22 there exists $U(t, s)$ satisfying (a)-(e). Let
\[
h := U(t, 0)h(0) + \int_0^t U(t, s)g(s)ds.
\] (111)

Then $h \in W^1_{3, \infty}$ and it is a solution of equation (110). Let $h_1, h_2$ be two solutions of equation (110) then $h_2 - h_1$ is a solution of
\[
\begin{aligned}
\dot{h} - A_0(t)h &= 0, \\
h|_{t=0} &= 0.
\end{aligned}
\] (112)

By Lemma 19 this equation has the unique solution $0$. Thus $h_2 - h_1 = 0$ and equation (110) has a solution $h \in W^1_{3, \infty}$ that is uniquely determined from $h(0) \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3$. The assertion follows. \[\square\]
Theorem 24. Let \( I := [0,T] \) (\( T > 0 \)). Let \( \nu > 0 \). Assume \( u,g \in W_3 \). Then the equation
\[
\dot{h} + \mathcal{P}((u \cdot \nabla) \mathcal{P}h + (\mathcal{P}h \cdot \nabla)u - \nu \Delta \mathcal{P}h) - g = 0
\]
has a solution \( h \in W_3 \) that is uniquely determined from \( h(0) \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3 \).

Proof. By Lemma 23, equation (113) has an unique solution \( h \) in \( W_3^{l,\infty} \). Since \( h \) satisfies equation (113), if \( h \in W_3^{l,\infty} \) for \( l \in \mathbb{N} \), it follows that \( \dot{h} \in W_3^{l,\infty} \), that is, \( h \in W_3^{l+1,\infty} \). Hence by induction it is proved that \( h \in W_3 \). The assertion follows. \( \square \)

6 Open maps between Fréchet spaces

Definition 25. Let \( E \) be a Hausdorff topological linear space. Assume the topology is defined by countably many seminorms on \( E \). Let \( d \) be a translation invariant complete metric on \( E \) induced by the seminorms. The pair \((E,d)\) is called a Fréchet space. We also denote the Fréchet space by \( E \).

Definition 26. Let \( E \) be a Fréchet space. A map \( \psi : E \to E \) is a contraction map if there exist \( 0 < K < 1 \) such that
\[
d(\psi(e_1),\psi(e_2)) \leq K \cdot d(e_1, e_2) \tag{114}
\]
for any \( e_1, e_2 \in E \).

Lemma 27. Let \( E \) be a Fréchet space. Then a contraction map \( \psi : E \to E \) has an unique fixed point.

Proof. We first show the existence. Let \( e \in E \). Consider \( \{\psi^N(e)\}_N \). By inequality (114) this is a Cauchy sequence with respect to \( d \). Thus since \( E \) is complete it is convergent. It is easy to show that the limit is a fixed point. The uniqueness follows from inequality (114). The assertion follows. \( \square \)

Lemma 28. Let \( E \) be a Fréchet space and let \( \Psi(e) := e + \psi(e) \ (e \in E) \), where \( \psi \) is a contraction map. Let \( r > 0 \) and \( 0 < s < 1 \). Assume that \( \psi(0) = 0 \) and that
\[
d(\psi(e_1),\psi(e_2)) < sr \tag{115}
\]
for all \( e_1, e_2 \in \{e \in E \mid d(0, e) < r\} \). If \( e' \in E \) and \( d(0, e') < (1-s)r \) then there exists an unique \( e \in \{e \in E \mid d(0, e) < r\} \) such that \( \Psi(e) = e' \).

Proof. Let \( e' \in E \) and \( d(0, e') < (1-s)r \). Let \( g_{e'}(e) := e' - \psi(e) \ (e \in E) \). Then by assumption \( g_{e'} \) is a contraction map on \( \{e \in E \mid d(0, e) < r\} \) and by Lemma 27 there exists an unique fixed point \( e \in \{e \in E \mid d(0, e) < r\} \). The assertion follows from this. \( \square \)
**Definition 29.** Let $\mathcal{X}$, $\mathcal{Z}$ be Fréchet spaces. Let $\Phi : \mathcal{X} \to \mathcal{Z}$ be a map. The Fréchet derivative $d\Phi(p)$ of $\Phi$ at $p \in \mathcal{X}$ is defined to be $A$ such that for any neighbourhoods $V$ and $W$ of $0 \in \mathcal{X}$ and $0 \in \mathcal{Z}$ and for any $q \in p + tV \ (t > 0)$

$$\Phi(q) - \Phi(p) - A(q - p) \in o(t)W \ (t \to 0).$$ (116)

$\Phi$ is a $C^1$-map if the Fréchet derivative $d\Phi(p)$ exists for any $p \in \mathcal{X}$ and the map $p \mapsto d\Phi(p)$, which is the Fréchet derivative of $\Phi$, is continuous.

**Lemma 30.** Let $\mathcal{X}$, $d_1$, $\mathcal{Z}$ be Fréchet spaces. A $C^1$-map $\Phi : \mathcal{X} \to \mathcal{Z}$ such that $d\Phi(p)$ is a topological linear isomorphism for each $p \in \mathcal{X}$ is an open mapping.

**Proof.** By assumption $d\Phi(p)$ is a topological linear isomorphism from $T_p\mathcal{X}$ to $T_{\Phi(p)}\mathcal{Z}$ for each $p \in \mathcal{X}$. Let $p \in \mathcal{X}$ and identify neighbourhoods $\mathcal{X}_p$ and $\mathcal{Z}_{\Phi(p)}$ of $p \in \mathcal{X}$ and $\Phi(p) \in \mathcal{Z}$ with the corresponding neighbourhoods $\mathcal{X}_p$ and $\mathcal{Z}_{\Phi(p)}$ of $0 \in T_p\mathcal{X}$ and $0 \in T_{\Phi(p)}\mathcal{Z}$. Then

$$(d\Phi(p))^{-1} \circ \Phi(q) = q + \varphi(q) \ (q \in \mathcal{X}_p),$$ (117)

where $\varphi$ is a contraction map of $\mathcal{X}_p$ and $d_1(0, \varphi(q)) = o(d_1(0, q))$. Let $r > 0$ be sufficiently small then by definition $\varphi(0) = 0$ and there exist $0 < s < 1$ such that

$$d_1(\varphi(e_1), \varphi(e_2)) < sr$$ (118)

for all $e_1, e_2 \in \{ e \in E \mid d_1(0, e) < r \}$. By Lemma 28

$$(d\Phi(p))^{-1} \circ \Phi(\{ e \in E \mid d_1(0, e) < r \}) \supset \{ e \in E \mid d_1(0, e) < (1 - s)r \}.$$ (119)

Here $d\Phi(p)$ is a topological isomorphism. Since $p \in \mathcal{X}$ is arbitrary it follows that $\Phi$ is an open mapping. The assertion follows. \(\square\)

For later use we give the following definition.

**Definition 31.** Let $\mathcal{X}$, $\mathcal{Z}$ be Fréchet spaces. Let $\Phi : \mathcal{X} \to \mathcal{Z}$ be a map. Let $k \in \mathbb{N}$. The Fréchet derivative $d^{(k)}\Phi$ of $\Phi$ of order $k$ is defined to be the Fréchet derivative of $d^{(k-1)}\Phi$. $\Phi$ is $C^\infty$ if the Fréchet derivative of $\Phi$ of any order exists and is continuous.

7 Navier-Stokes equations on $\mathbb{R}^3/\mathbb{Z}^3$

We prove Theorem 2. Let $\nu > 0$ and $I := [0, T]$ ($T > 0$). Let

$$\Phi : W_3 \times (W_1)_{\nabla} \to W_3 \times (W_3)_{\text{div}} \times C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3.$$ (120)
be given by
\[(u, P) \mapsto \begin{bmatrix}
\dot{u} + (u \cdot \nabla)u - \nu \Delta u + P \\
\nabla \cdot u \\
u(0)
\end{bmatrix}, \quad (121)
\]
where
\[(W_1)_\nabla := \{ v \mid v = \nabla w (\exists w \in W_1) \}, \quad (122)
\]
and
\[(W_3)_{\text{div}} := \{ v \mid v = \nabla \cdot w (\exists w \in W_3) \}. \quad (123)
\]
We introduce to \((W_1)_\nabla\) the relative topology as a subset of \(W_3\) and to \((W_3)_{\text{div}}\) the relative topology as a subset of \(W_1\). Then \(\Phi\) is \(C^\infty\) in \((u, P)\) in the sense of Fréchet derivatives (see Definition 31). We begin with proving the following lemma:

**Lemma 32.** The map
\[
d\Phi((u, P)) : (h, \beta) \mapsto \begin{bmatrix}
\dot{h} + (u \cdot \nabla)h + (h \cdot \nabla)u - \nu \Delta h + \beta \\
\nabla \cdot h \\
h(0)
\end{bmatrix} \quad (124)
\]
is a linear isomorphism from
\[W_3 \times (W_1)_\nabla \quad (125)\]
to
\[
\{ (a, b, c) \in W_3 \times (W_3)_{\text{div}} \times C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3 \mid \nabla \cdot c = b \}. \quad (126)
\]

**Proof.** By calculation \(d\Phi\) is given by the map (124). Let
\[\{ (a, b, c) \in W_3 \times (W_3)_{\text{div}} \times C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3 \mid \nabla \cdot c = b \}. \quad (127)\]
Let
\[
\begin{align*}
\dot{h} + (u \cdot \nabla)h + (h \cdot \nabla)u - \nu \Delta h + \beta &= a, \\
\nabla \cdot h &= b, \\
h(0) &= c.
\end{align*} \quad (128-130)
\]
From equation (129), \(h \in W_3\) is determined up to \(\text{Ker}(\nabla \cdot (\cdot))\). Since \(\beta \in (W_1)_\nabla\), by Theorem 17, Theorem 24 and an elementary argument \(h \in W_3\) is uniquely determined from equation (128) and equation (130). Then from equation (128), \(\beta \in (W_1)_\nabla\) is uniquely determined. From this the assertion follows. \(\square\)

Let
\[\mathcal{X} := W_3 \times (W_1)_\nabla \quad (131)\]
and
\[ \mathcal{Z} := \{ r := (f, r_2, u_0) \in W_3 \times (W_3)_{\text{div}} \times C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3 \mid \nabla \cdot u_0 = r_2(0) \}. \]

Then \( \Phi \) is a map from \( \mathcal{Y} \) to \( \mathcal{Z} \). Let
\[ \mathcal{Y} := \Phi(\mathcal{X}). \]

Let \( p = (u, P) \in \mathcal{X} \) be an arbitrary point.

**Lemma 33.** \( d\Phi(p) \) is a topological isomorphism.

**Proof.** Note that \( \Phi \) is a \( C^1 \)-map from \( \mathcal{X} \) to \( \mathcal{Z} \) and from Lemma 32
\[
d\Phi(p) : (h, \beta) \mapsto \begin{bmatrix} h + (u \cdot \nabla)h + (h \cdot \nabla)u - \mu \Delta h + \beta \\
\nabla \cdot h \\
h(0) \end{bmatrix}
\]
is a linear isomorphism and by the open mapping principle a topological linear isomorphism from
\[ T_p \mathcal{X} = \{(h, \beta) \in W_3 \times (W_1)_{\nabla} \} \]
to
\[ T_{\Phi(p)} \mathcal{Z} = \{(a, b, c) \in W_3 \times (W_3)_{\text{div}} \times C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3 \mid \nabla \cdot c = b(0) \}. \]
The assertion follows. \[ \square \]

**Lemma 34.** \( \mathcal{Y} \) is a Fréchet manifold.

**Proof.** By Lemma 33 \( d\Phi(p) \) is a topological isomorphism so that, since \( p \) is arbitrary, by Lemma 30 \( \Phi : \mathcal{X} \to \mathcal{Z} \) is an open map. Since \( \mathcal{Y} = \Phi(\mathcal{X}) \) it is a Fréchet manifold. The assertion follows. \[ \square \]

Let
\[ \varphi_p : \mathcal{X} \xrightarrow{\sim} T_p \mathcal{X} \]
and
\[ \psi_{\Phi(p)} : \mathcal{X} \xrightarrow{\sim} T_{\Phi(p)} \mathcal{Y} \]
be canonical isomorphisms. Let
\[ U_p := \varphi_p^{-1}(T_p \mathcal{X})(= \mathcal{X}) \]
and
\[ V_{\Phi(p)} := \psi_{\Phi(p)}^{-1}(T_{\Phi(p)} \mathcal{Y}). \]
Observe that by Lemma 33

$$d\Phi(p) : T_p \mathcal{X} \xrightarrow{\cong} T_{\Phi(p)} \mathcal{Y}$$  \hspace{1cm} (141)$$

is a topological linear isomorphism. Let \( k \in \mathbb{N} \). Let

$$(W_3^{k,k+2})_{\text{div}} := \{ v \mid v = \nabla \cdot w \ (\exists w \in W_3^{k,k+2}) \}. \hspace{1cm} (142)$$

Introduce to \((W_3^{k,k+2})_{\text{div}}\), the relative topology as a subset of \( W_1^{k,k+1} \) and to \( \mathcal{Y} \) the relative topology as a subset of

$$W_3^{k,k} \times (W_3^{k,k+2})_{\text{div}} \times H^{k+2}(\mathbb{R}^3/\mathbb{Z}^3)^3, \hspace{1cm} (143)$$

which defines a seminorm \( p_k \) (it is in fact a norm) of \( \mathcal{Y} \). Let the system of open sets of the topology of \( \mathcal{Y} \) be

$$\{ \mathcal{O} \}. \hspace{1cm} (144)$$

Introduce to \( T_{\Phi(p)} \mathcal{Y} \) the topology induced from \( \theta_{\Phi(p)} := (\psi_{\Phi(p)})^{-1}|_{T_{\Phi(p)} \mathcal{Y}} \), i.e.

$$\{ \theta_{\Phi(p)}^{-1}(\mathcal{O}) \}, \hspace{1cm} (145)$$

to \( T_p \mathcal{X} \) the topology induced from \( \theta_{\Phi(p)} \circ d\Phi(p) \), i.e.

$$\{ (d\Phi(p))^{-1} \circ \theta_{\Phi(p)}^{-1}(\mathcal{O}) \}, \hspace{1cm} (146)$$

to \( \mathcal{U}_p \) the topology induced from \( \theta_{\Phi(p)} \circ d\Phi(p) \circ \varphi_p|\mathcal{U}_p \), i.e.

$$\{ (\varphi_p|\mathcal{U}_p)^{-1} \circ (d\Phi(p))^{-1} \circ \theta_{\Phi(p)}^{-1}(\mathcal{O}) \}, \hspace{1cm} (147)$$

and to \( \mathcal{V}_{\Phi(p)} \) the topology induced from \( \theta_{\Phi(p)} \circ \psi_{\Phi(p)}|\mathcal{V}_{\Phi(p)} \), i.e.

$$\{ (\psi_{\Phi(p)}|\mathcal{V}_{\Phi(p)})^{-1} \circ \theta_{\Phi(p)}^{-1}(\mathcal{O}) \}. \hspace{1cm} (148)$$

**Lemma 35.** \( \mathcal{U}_p = \mathcal{X} \) and \( \mathcal{V}_{\Phi(p)} = \mathcal{X} \).

*Proof.* From the definitions, using the canonical isomorphisms (137) and (138), the assertion follows. \( \square \)

**Remark 36.** \( \Phi \) satisfies Navier-Stokes condition 1. That \( p_k \) is a norm is Navier-Stokes condition 2. (See Remark 65.)

By Lemma 35 \( \mathcal{U}_p = \mathcal{X} \) and \( \mathcal{V}_{\Phi(p)} = \mathcal{X} \). The topologies are induced by a seminorm \( p_k \) such that \( p_k(r) \) implies \( r = 0 \) (for \( r \in \mathcal{Y} \)), \( q \mapsto d\Phi(q) \) is continuous with respect to the ordinary topology of \( \mathcal{U}_p \) and the topology of \( T_{\Phi(p)} \mathcal{V}_{\Phi(p)}(= T_{\Phi(p)} \mathcal{Y}) \) induced by \( p_k \).
Definition 37. Let $Y$ be a Banach manifold and $B$ a Banach space. Let $L(q) : T_qY \to B$ ($q \in Y$) be a map. Then the set of accumulation points of the sequences

$$\sum_i L(\gamma(\xi_i^j))\gamma(\xi_i^j)|E_i^j|,$$

(149)

as $j \to \infty$, where $\gamma : [0,1] \to Y$ runs over all smooth paths from $p \in Y$ to $q \in Y$, $\{E_i^j\}$ all sequences of measurable sets of $[0,1]$ such that $\prod_i E_i^j = [0,1]$ and $\sup\limits_i |E_i^j| \to 0$ ($j \to \infty$) and $\xi_i^j$ all elements of $E_i^j$, if exists, is denoted by $\int_p^q L(q')$.

Remark 38. In general $q \mapsto \int_p^q L(q')$ is a multi-valued map.

The following is a result of [16], of which expression is slightly changed.

Theorem 39 (M. A. Rieffel). Let $(X,S,\mu)$ be a $\sigma$-finite positive measure space and let $B$ be a Banach space. Then $m$ is the indefinite integral with respect to $\mu$ of a $B$-valued Bochner integrable function on $X$ if and only if

1. $m(E) = 0$ whenever $\mu(E) = 0$, $E \in S$,
2. the total variation $|m|$ of $m$ is a finite measure,
3. given $E \in S$ with $0 < \mu(E) < \infty$ there exists an $F \subset E$ such that $\mu(F) > 0$

and

$$A_F(m) := \left\{ \frac{m(F')}{\mu(F')} \mid F' \subset F, \mu(F') > 0 \right\}$$

(150)

is relatively (norm) compact.

Definition 40. A topological manifold $Y$ is a manifold with a topological linear space $T_qY$ for each $q \in Y$ such that there exists for each $q \in Y$ a homeomorphism $\theta_q$ from a neighbourhood $W$ of $q$ to a neighbourhood $W'$ of $0 \in T_qY$. A topological manifold $Y$ is parametrized if for each $q \in Y$ there exists a fundamental system $\{W'_\tau\}_{\tau}$ of neighbourhoods of $0 \in T_qY$ decreasing as $\tau \to 0$. A $C^1$-map between parametrized topological manifolds $Y_1, Y_2$ with fundamental systems $\{W'_1, \tau\}, \{W'_2, \tau\}$ of neighbourhoods is a map $\xi$ with the following property: for each $q \in Y_1$ there exists $d\xi(q)$ such that for small $\tau$ and for $q' - q \in W'_1, \tau$

$$\xi(q') - \xi(q) - d\xi(q)(q' - q) \in W'_2, \alpha(\tau),$$

(151)

where a specified neighbourhood $W_1$ of $q \in Y_1$ and a specified neighbourhood $W'_1$ of $0 \in T_qY_1$ are identified. $d\xi(q)$ is the Fréchet derivative of $\xi$ at $q$. A (not necessarily $C^1$-)map $\xi : Y_1 \to Y_2$ has derivative a.e. along a smooth path $\gamma$ if $d(\xi(\gamma(t)))$ ($t \in [0,1]$) exists a.e. for a smooth path $\gamma : [0,1] \to Y_1$.

The following is a consequence of [4], Chapter 5, Section 5.7, Theorem 1 (Rellich-Kondrachov Compactness Theorem).
**Theorem 41.** Let \( k \in \mathbb{Z}_{\geq 0} \). Let \( \Omega \subset \mathbb{R}^3 \) be a bounded open domain. Let \( H^k(\Omega) \) be the Sobolev space. Then the inclusion \( H^{k+1}(\Omega) \hookrightarrow H^k(\Omega) \) is a compact operator.

**Lemma 42.** Let \( \mathcal{X}_p \) be a sufficiently small convex neighbourhood of \( p \in \mathcal{X} \) identified with the corresponding neighbourhood of \( 0 \in T_q \mathcal{X}_p \) \((q \in \mathcal{X}_p)\). Let \( \{q_j\} \subset \mathcal{X}_p \) be a sequence. Then there exists a subsequence of \( \{q_j\} \) convergent to \( 0 \in \mathcal{X}_p \) with respect to the ordinary topology.

**Proof.** Let \( k_1, k_2 \in \mathbb{Z}_{\geq 0} \). Let \((u_j, P_j) := q_j\). By Theorem 41 possibly passing to a subsequence \((\partial_t^{k_1} u_j(t), \partial_t^{k_1} P_j(t))\) is convergent for fixed \( t \in I \). Let 

\[
(\partial_t^{k_1} u(t), \partial_t^{k_1} P(t)) := \lim_{j \to \infty} (\partial_t^{k_1} u_j(t), \partial_t^{k_1} P_j(t)).
\]

By diagonal argument the same formula holds for any \( t_0 \in I' \), where \( I' \) is a dense countable subset of \( I \). By assumption \( ||(\partial_t^{k_1+1} u_j, \partial_t^{k_1+1} P_j)||_{W^{0,0\times(0,1)}(\mathbb{R}^3)} \) is bounded on \( I \) so that \((\partial_t^{k_1} \partial_x^2 u(t_0), \partial_t^{k_1} \partial_x^2 \nabla p(t_0)) \) \((t_0 \in I')\) extends to a continuous function on \( t \in I \). The resulting function is denoted by the same symbol. Since \(||(\partial_t^{k_1+1} u_j, \partial_t^{k_1+1} P_j)||_{W^{0,0\times(0,1)}(\mathbb{R}^3)} \) is bounded there exists \( M > 0 \) such that 

\[
||(\partial_t^{k_1} u_j(t), \partial_t^{k_1} P_j(t)) - (\partial_t^{k_1} u_j(t_0), \partial_t^{k_1} P_j(t_0))||_{W^{0,k_2 \times (0,1)}} \leq M|t - t_0| \tag{153}
\]

and 

\[
||(\partial_t^{k_1} u(t), \partial_t^{k_1} P(t)) - (\partial_t^{k_1} u(t_0), \partial_t^{k_1} P(t_0))||_{W^{0,k_2 \times (0,1)}} \leq M|t - t_0|. \tag{154}
\]

For any \( t \in I \) and any \( \epsilon > 0 \) there exists \( t_0 \in A \) such that \( M|t - t_0| < \frac{\epsilon}{3} \) so that by above for large \( j \)

\[
||(\partial_t^{k_1} u_j(t), \partial_t^{k_1} P_j(t)) - (\partial_t^{k_1} u(t), \partial_t^{k_1} P(t))||_{W^{0,k_2 \times (0,1)}} \leq ||(\partial_t^{k_1} u_j(t), \partial_t^{k_1} P_j(t)) - (\partial_t^{k_1} u_j(t_0), \partial_t^{k_1} P_j(t_0))||_{W^{0,k_2 \times (0,1)}} + ||(\partial_t^{k_1} u_j(t_0), \partial_t^{k_1} P_j(t_0)) - (\partial_t^{k_1} u(t_0), \partial_t^{k_1} P(t_0))||_{W^{0,k_2 \times (0,1)}} + ||(\partial_t^{k_1} u(t_0), \partial_t^{k_1} P(t_0)) - (\partial_t^{k_1} u(t), \partial_t^{k_1} P(t))||_{W^{0,k_2 \times (0,1)}} \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \tag{161}
\]

Thus possibly passing to a subsequence \((\partial_t^{k_1} u_j(t), \partial_t^{k_1} P_j(t)) \to (\partial_t^{k_1} u(t), \partial_t^{k_1} P(t)) \) \((t \in I)\). Since \( \mathbb{R}^3 / \mathbb{Z}^3 \) is compact the above formula also shows that the convergence is uniform. Since \( k_1, k_2 \) are arbitrary the assertion follows. \(\square\)

The set \( X \) equipped with the topology induced by \( p_t \) is, if exists, denoted by \( X^w \).
Lemma 43. Let $\mathcal{X}_p$ be a sufficiently small convex neighbourhood of $p \in \mathcal{U}_p$ with respect to the ordinary topology identified with the corresponding neighbourhood of $0 \in T_q \mathcal{U}_p := T_q \mathcal{X}_p$ ($q \in \mathcal{X}_p$). Then for any sequence $\{q_j\} \subset \mathcal{X}_p^w$ convergent to $0 \in \mathcal{X}_p^w$ is convergent to $0 \in \mathcal{X}_p$ with respect to the ordinary topology.

Proof. Observe that $C^\infty(\mathbb{R}^3/\mathbb{Z}^3) = H^\infty(\mathbb{R}^3/\mathbb{Z}^3)$. Assume possibly after passing to a subsequence that $q_j \notin c\mathcal{X}_p$ ($0 < c < 1$) for large $j$. $\mathcal{X}_p$ is sufficiently small so that by Lemma 42 there exists a subsequence $\{q_j\}$ convergent with respect to the ordinary topology. Let $W^1, \ldots, W^m$ be neighbourhoods of $p \in \mathcal{U}_p$ with respect to the topology induced by $p_k$ and $\bar{W}^1, \ldots, \bar{W}^m$ their closures with respect to this topology. Let $\mathcal{X}_p^w$ be the closure of $\mathcal{X}_p$ with respect to the ordinary topology. Then since $q_j \in \bar{W}^1 \cap \cdots \cap \bar{W}^m \cap (\mathcal{X}_p^w \setminus c\mathcal{X}_p^w)$ for large $l$ and $W^1, \ldots, W^m$ are arbitrary and since from above ($\mathcal{X}_p$ and so $\mathcal{X}_p^w \setminus c\mathcal{X}_p^w$ is compact with respect to (the ordinary topology and so) the topology induced by $p_k$ an elementary argument shows that

$$\bigcap_{W}(\bar{W} \setminus c\mathcal{X}_p^w) \neq \emptyset. \quad (162)$$

This contradicts with the definition of the topology of $\mathcal{U}_p$ induced by $p_k$. Thus $q_j \in c\mathcal{X}_p$ for large $j$. Since $c$ is arbitrary $\{q_j\} \subset \mathcal{X}_p$ is convergent to $0 \in \mathcal{X}_p$ with respect to the ordinary topology. The assertion follows.

The integral with respect to the topologies induced by $p_k$ (resp. with respect to the ordinary topology of $\mathcal{U}_p$ and the topology of $\mathcal{V}_{\Phi(p)}$ induced by $p_k$) is denoted by $\int$ (resp. $(\int)^*$). See Definition 37.

Lemma 44. There exist a sufficiently small convex neighbourhood $\mathcal{X}_p$ of $p \in \mathcal{U}_p$ with respect to the ordinary topology and a sufficiently small neighbourhood of $0 \in (T_{\Phi(p)}\mathcal{Y})^w$ identified with a sufficiently small neighbourhood $\mathcal{Y}_{\Phi(p)}^w$ of $\Phi(p) \in \mathcal{V}_{\Phi(p)}^w$ such that the multi-valued map $q \in \mathcal{X}_p^w \mapsto \int_0^1 dq \Phi(q') \in \mathcal{Y}_{\Phi(p)}^w$ has a continuous branch.

Proof. Since $\mathcal{X}_p$ is sufficiently small the map $d\Phi(q)$ ($q \in \mathcal{X}_p$) from $T_{\Phi(p)}\mathcal{X}_p$ to $\mathcal{Y}_{\Phi(p)}^w$ is continuous and the function $q \mapsto ((\int_0^1)^*d\Phi(q'))\llcorner_{(T_{\Phi(p)}\mathcal{Y}_{\Phi(p)})^w}$ is well-defined. Let $\gamma_q(t) := (1 - t)p + tq$ ($q \in \mathcal{X}_p$). Since $d\Phi$ is well-defined and a continuous map from $\mathcal{X}_p$ to $\mathcal{Y}_{\Phi(p)}^w$ there exists $M > 0$ such that

$$\lim_{h \to 0} ||\frac{1}{h} \left( \int_{\gamma_q(t_1 + h)}^{\gamma_q(t_1)} dq \Phi(q') \right)\llcorner_{(T_{\Phi(p)}\mathcal{Y}_{\Phi(p)})^w} || \leq M. \quad (163)$$

$$= \lim_{h \to 0} \left( \int_0^1 \left( dq \Phi((1 - t)\gamma_q(t_1) + t\gamma_q(t_1 + h)) - \gamma_q(t_1 + h) - \gamma_q(t_1) \right) dt \right) \llcorner_{(T_{\Phi(p)}\mathcal{Y}_{\Phi(p)})^w} \quad (164)$$

$$= \lim_{h \to 0} ||\left( \int_0^1 \left( dq \Phi((1 - t)\gamma_q(t_1) + t\gamma_q(t_1 + h)) - \gamma_q(t_1 + h) - \gamma_q(t_1) \right) dt \right) \llcorner_{(T_{\Phi(p)}\mathcal{Y}_{\Phi(p)})^w} || \leq M. \quad (165)$$
By Theorem 39 there exists an integrable function $S_{\gamma}$ (see Definition 37) such that
\[
\left( \int_{\gamma(t_1)}^{\gamma(t_2)} \right)\Phi(q') = \int_{t_1}^{t_2} S_{\gamma} dt,
\]
where $0 \leq t_1 \leq t_2 \leq 1$. Then by Lemma 43 an easy argument shows that the map $q \in \mathcal{X}_p^w \mapsto \int_0^1 S_{\gamma}(t) dt \in \mathcal{Y}_{\Phi(p)}^w$ gives a desired branch. The assertion follows.

**Lemma 45.** There exist a sufficiently small convex neighbourood $\mathcal{X}'_p$ of $p \in \mathcal{U}_p$ with respect to the ordinary topology and a sufficiently small neighbourhood of $0 \in (T_{\Phi(p)}^w)_{\mathcal{Y}}^w$ identified with a sufficiently small neighbourhood $\mathcal{Y}_{\Phi(p)}^w$ of $\Phi(p) \in V_{\Phi(p)}^w$ such that
\[
\Phi(q) = \Phi(p) + \int_p^q d\Phi(q') \quad (q \in \mathcal{X}_p^w, \Phi(q) \in \mathcal{Y}_{\Phi(p)}^w)
\]
holds for a well-defined branch of $q \in \mathcal{X}_p^w \mapsto \int_p^q d\Phi(q') \in \mathcal{Y}_{\Phi(p)}^w$.

**Proof.** By Lemma 44 the multi-valued map $q \in \mathcal{X}_p^w \mapsto \int_p^q d\Phi(q') \in \mathcal{Y}_{\Phi(p)}^w$ has a continuous branch so that the function
\[
\Phi'(q) := \Phi(p) + \int_p^q d\Phi(q')
\]
has a well-defined branch. Note that since $\Phi'$ has the derivative a.e. along $\gamma_q$ and since the topology of $\mathcal{X}_p^w$ induced by $p_k$ is weaker than the ordinary one it is obtained that (*) the derivatives a.e. along $\gamma_q$ of $\Phi'$ with respect to the both topologies at $q$ are equal to $d\Phi(q)$ a.e. in the path. On the other hand it is true that
\[
\Phi(q) = \Phi(p) + \left( \int_p^q \right)^* d\Phi(q').
\]
Then $\Phi$ and $\Phi'$ satisfy the same conditions: (i) $\Xi(p) = \Phi(p)$ and (ii) the derivative of $\Xi(q)$ a.e. along $\gamma_q$ with respect to the ordinary topology of $\mathcal{X}_p$ and the topology of $\mathcal{Y}_{\Phi(p)}^w$ induced by $p_k$ coincides with $d\Phi(q)$ a.e. in the path. It follows that $\Phi(q) = \Phi'(q)$. By (*) it is obtained for $t \in [0, 1]$ that
\[
S_{\gamma}(t) = d\Phi(\gamma_q(t)) \gamma_q(t)
\]
(if LHS exists and is equal to $d(\Phi'(\gamma_q(t)))$). Thus for this branch
\[
\Phi(q) = \Phi(p) + \int_p^q d\Phi(q').
\]
The assertion follows.
\[ \mathcal{X}_p \text{ and } \mathcal{X}_p^2 \text{ are identified with the corresponding neighbourhoods of } 0 \in T_{\mathcal{X}_p} (q \in \mathcal{X}_p) \text{ and } 0 \in T_{Q} T_{\mathcal{X}_p} (Q \in \mathcal{X}_p^2). \]

**Lemma 46.** Shrink \( \mathcal{X}_p \) if necessary. Then \( d\Phi : (\mathcal{X}_p^2)^w \to (TT\mathcal{Y}_{\Phi(p)})^w \) is continuous and the Fréchet derivative of \( \Phi : \mathcal{X}^w \to \mathcal{Y}_{\Phi(p)}^w \) (see Definition 40) is equal to \( d\Phi \) on \( (\mathcal{X}_p^2)^w \).

**Proof.** By Lemma 43 it is proved that \( d\Phi \) is continuous on \( (\mathcal{X}_p^2)^w \). From this and Lemma 45 the assertion follows easily. \( \square \)

**Lemma 47.** Shrink \( \mathcal{X}_p \) if necessary. Then \( d(d\Phi) : (\mathcal{X}_p^2)^w \to (TT\mathcal{Y}_{\Phi(p)})^w \) is continuous and the Fréchet derivative of \( d\Phi : (\mathcal{X}_p^2)^w \to (TT\mathcal{Y}_{\Phi(p)})^w \) (see Definition 40) is equal to \( d(d\Phi) \) on \( (\mathcal{X}_p^2)^w \).

**Proof.** This is proved as in Lemma 46. \( \square \)

**Lemma 48.** Let \( L(T_{\mathcal{X}_p} \mathcal{X}_p^w \to (TT\mathcal{Y}_{\Phi(p)})^w \) be the set of continuous linear operators from \( (T_{\mathcal{X}_p} \mathcal{X}_p^w \to (TT\mathcal{Y}_{\Phi(p)})^w \). Then there exists \( M' > 0 \) such that

\[
||d\Phi(q') - d\Phi(q)||L((T_{\mathcal{X}_p} \mathcal{X}_p^w \to (TT\mathcal{Y}_{\Phi(p)})^w) \leq M'||q' - q||_{\mathcal{X}_p^w},
\]

(173)

for \( q, q' \in \mathcal{X}_p^w \), and such that

\[
||\Phi(q') - \Phi(q) - d\Phi(q)(q' - q)||_{\mathcal{Y}_{\Phi(p)}^w} \leq M'||q' - q||^2_{\mathcal{X}_p^w},
\]

(174)

for \( q, q' \in \mathcal{X}_p^w \).

**Proof.** By Lemma 47 \( d(d\Phi) : (\mathcal{X}_p^2)^w \to (TT\mathcal{Y}_{\Phi(p)})^w \) is continuous. Shrinking \( \mathcal{X}_p \) if necessary there exists \( M' > 0 \) such that for \( Q \in (\mathcal{X}_p^2)^w \)

\[
||d(d\Phi)(Q)|| \leq M'.
\]

(175)

Let \( \gamma : [0, 1] \to \mathcal{X}_p^w \) be a smooth path from \( q \) to \( q' \). Then, by Lemma 47

\[
||d\Phi(q') - d\Phi(q)||L((T_{\mathcal{X}_p} \mathcal{X}_p^w \to (TT\mathcal{Y}_{\Phi(p)})^w) \leq ||\int_0^1 \frac{\partial}{\partial t} d\Phi(\gamma(t))dt||L((T_{\mathcal{X}_p} \mathcal{X}_p^w \to (TT\mathcal{Y}_{\Phi(p)})^w)
\]

(176)

\[
\leq M'||q' - q||_{\mathcal{X}_p^w},
\]

(177)

and

\[
||\Phi(q') - \Phi(q) - d\Phi(q)(q' - q)||_{\mathcal{Y}_{\Phi(p)}^w}
\]

(178)

\[
= ||\int_0^1 \frac{\partial^2}{\partial t^2} \Phi(\gamma(t))dt||_{\mathcal{Y}_{\Phi(p)}^w}
\]

(179)

\[
\leq M'||q' - q||^2_{\mathcal{X}_p^w}.
\]

(180)

The assertion follows. \( \square \)
Take the completions $\mathcal{U}^k_p$ and $\mathcal{V}^k_{\Phi(p)}$ of $\mathcal{X}^w_p$ and $\mathcal{Y}^w_{\Phi(p)}$ induced from $p_k$. By Lemma 48, $d\phi$ extends to a continuous map, $\Phi$ extends to a $C^1$-map from $\mathcal{U}^k_p$ to $\mathcal{V}^k_{\Phi(p)}$ and the Fréchet derivative with respect to the topologies of $\mathcal{U}^k_p$ and $\mathcal{V}^k_{\Phi(p)}$ of the extended $\Phi$ at $q$ is equal to the value $d\phi(q)$ at $q$ of the extended $d\phi$. $d\phi(p)$ extends to a topological isomorphism between the completions $T_p\mathcal{U}^k_p$ and $T_{\Phi(p)}\mathcal{V}^k_{\Phi(p)}$ of $(T_p\mathcal{X})^w$ and $(T_{\Phi(p)}\mathcal{Y})^w$ induced from $p_k$ so that since $\mathcal{X}^w_p$ is sufficiently small $d\phi(q)$ ($q \in \mathcal{U}^k_p$) is a topological isomorphism. By Theorem 12 there exist sufficiently small neighbourhoods $U_p$ and $V_{\Phi(p)}$ of $p \in \mathcal{U}^k_p$ and $\Phi(p) \in \mathcal{V}^k_{\Phi(p)}$ such that the extended

$$\Phi : U_p \xrightarrow{\cong} V_{\Phi(p)} \quad (182)$$

is an isomorphism.

**Remark 49.** We have taken $O = \mathcal{U}^k_p$, $X$ the linear hull of $\mathcal{U}^k_p$, $Y$ the linear hull of $\mathcal{V}^k_{\Phi(p)}$, $\xi = \Phi$ and $a_0 = p$.

Observe that $V_{\Phi(p)}$ is an open set of the completion of $\mathcal{X}$. We thus obtain:

**Lemma 50.** $V_{\Phi(p)}$ ($p \in \mathcal{X}$) form a Banach manifold.

Replacing $I$ with an interval $J := [t_0, t_0 + T_0]$ ($0 \leq t_0 < T$ and $T_0 > 0$ is small) we define $\mathcal{X}^J$, $\mathcal{Y}^J$, $U^J_p$, $\Phi^J$ etc. in the same way as $\mathcal{X}$, $\mathcal{Y}$, $U_p$, $\Phi$ etc. The above two (1 and 2) of Navier-Stokes conditions also hold if we replace $I$ with $J$ (see Remark 65).

**Remark 51.** Navier-Stokes condition 3 (see Remark 65) is satisfied.

We need a lemma.

**Lemma 52.** If $U^J_{p1} \cap U^J_{p2} \neq \emptyset$, it is open in $U^J_{p1}$.

**Proof.** Let $p \in U^J_{p1} \cap U^J_{p2} \cap \mathcal{X}^J$. The tangent space $T_pU^J_{p1}$ of $U^J_{p1}$ at $p$ and the tangent space $T_pU^J_{p2}$ of $U^J_{p2}$ at $p$ are by definition given as the completions of $T_p\mathcal{X}^J$ and $\Phi^J$ is a local diffeomorphism on $U^J_{p1}$ and $U^J_{p2}$. Thus $T_pU^J_{p1} = T_pU^J_{p2} = (d\Phi^J(p))^{-1}(\text{completion of } T_{\Phi^J(p)}\mathcal{Y}^J)$, where the completion of $T_{\Phi^J(p)}\mathcal{Y}^J$ is induced from $\mathcal{Y}^J \subset \bigcup_p V^J_{\Phi^J(p)}$. There exists a canonical isomorphism $\varphi^J_p|_W$ from a neighbourhood $W$ of $p \in \mathcal{X}^J$ to the corresponding neighbourhood $W'$ of $0 \in T_p\mathcal{X}^J$ and $\mathcal{U}^k_p \cup \mathcal{U}^k_{p2}$ is the completion of $\mathcal{X}^J \cup \mathcal{X}^J$ induced from $p_k$. Extend $\varphi^J_p|_W$ to $W^1$ (resp. $W^2$), where $W^1$ (resp. $W^2$) is the closure of $W$ in $U^J_{p1}$ (resp. $U^J_{p2}$), to obtain a homeomorphism $\varphi^J_{p,1}$ (resp. $\varphi^J_{p,2}$). By definition $\varphi^J_{p,1}(W^1)$ (resp. $\varphi^J_{p,2}(W^2)$) is the closure of $W'$ in $T_pU^J_{p1}$ (resp. $T_pU^J_{p2}$). Since $T_pU^J_{p1} = T_pU^J_{p2}$ it follows that $\varphi^J_{p,1}(W^1) = \varphi^J_{p,2}(W^2)$ and since $U^J_{p1} \cup U^J_{p2} \subset \mathcal{U}^k_{p1} \cup \mathcal{U}^k_{p2}$, from the above, an easy argument shows that $(\varphi^J_{p,1})^{-1} = (\varphi^J_{p,2})^{-1}$. In particular $(\varphi^J_{p,2})^{-1}(W^2)$ is open in $U^J_{p1}$. By the definition of the topology $U^J_{p1} \cap U^J_{p2} \cap \mathcal{X}^J$ is dense in $U^J_{p1} \cap U^J_{p2}$. 27
Let \( p' \in U^J_{p_1} \cap U^J_{p_2} \) be an inner point of \( U^J_{p_2} \). Take a small neighbourhood \( N \) of \( p' \) in \( U^J_{p_2} \) and consider \( U^J_{p_1} \cap N \cap X^J \), which is by the same argument as above open in \( U^J_{p_1} \cap X^J \), that is, of the form \( O \cap U^J_{p_1} \cap X^J \), where \( O \) is an open set in \( U^J_{p_1} \). Take the closure of \( U^J_{p_1} \cap N \cap X^J \). Then the set of all inner points of this closure is open in \( U^J_{p_1} \) and the boundary of the closure does not contain \( p' \). Hence \( U^J_{p_1} \cap U^J_{p_2} \) is open in \( U^J_{p_1} \). The assertion follows.

**Corollary 53.** \( U^J_p \ (p \in X^J) \) form a Banach manifold.

We shall prove the local existence and uniqueness of a sufficiently smooth solution of equation (7) (of which smoothness depends on \( k \in \mathbb{N} \)).

Let \( t_0 \in I \). Introduce to the inductive limits \( \mathcal{U}_{t_0} \) of \( \bigcup_p U^J_p \) for \( J \ni t_0 \) and \( \mathcal{C}_{t_0} \) of

\[
\mathcal{X}^{(k),J} := \{(f,r_2,u_0) \in (W^{k,k})^J \times (((W^{k,k+2})_{\text{div}})^J \times H^{k+2}(\mathbb{R}^3/\mathbb{Z}^3))^3 \ (183) \\
| \nabla \cdot u_0 = r_2(t_0) \} (184)
\]

for \( J \ni t_0 \) the natural topologies (the quotient topologies induced from the maps \( \prod_j U^J_p \to \mathcal{U}_{t_0} \) and \( \prod_j \mathcal{X}^{(k),J} \to \mathcal{C}_{t_0} \)). We first prove the following lemma.

**Lemma 54.** Let \( r := (f,r_2,u_0) \in \mathcal{X}^J \). There exists \( (u, \nabla p) \in X^J \) such that

\[
\left\{ \begin{array}{l}
(\dot{u} + (u \cdot \nabla)u - \nu \Delta u + \nabla p)|_{t=t_0} = f(t_0), \\
(\nabla \cdot u)|_{t=t_0} = r_2(t_0), \\
u|_{t=t_0} = u_0, \\
(\nabla \cdot \dot{u})|_{t=t_0} = \dot{r}_2(t_0).
\end{array} \right. (185)
\]

**Proof.** Since

\[
(\dot{u} + (u \cdot \nabla)u - \nu \Delta u + \nabla p)|_{t=t_0} = f(t_0), (186)
\]

by assumption

\[
-\dot{u}|_{t=t_0} - \nabla p(t_0) = (u_0 \cdot \nabla)u_0 - \nu \Delta u_0 - f(t_0). (187)
\]

By \( (\nabla \cdot \dot{u})|_{t=t_0} = \dot{r}_2(t_0) \), \( -\dot{u}|_{t=t_0} \) is determined up to \( \text{Ker}(\nabla \cdot (\cdot)) \). Then by Theorem 17 it is confirmed that there exists such \( (u, \nabla p) \). The assertion follows.

\[
\{\Phi^J\}_J \text{ induces a map } \hat{\Phi}_{t_0} : \mathcal{U}_{t_0} \to \mathcal{C}_{t_0}.
\]

**Lemma 55.** The induced map \( \hat{\Phi}_{t_0} : \mathcal{U}_{t_0} \to \mathcal{C}_{t_0} \) is a homeomorphism.

28
Proof. \( \Phi^J : \bigcup \mathcal{U}_p^J \to \bigcup \mathcal{V}_p^J \) is a local diffeomorphism so that by definition \( \tilde{\Phi}_{t_0} : \mathcal{X}_{t_0} \to \mathcal{C}_{t_0} \) is a local homeomorphism. \( \mathcal{X}_{t_0} \) is connected (since \( \mathcal{X}^J \) is connected and each connected component of \( \mathcal{U}_p^J \) intersects with \( \mathcal{X}^J \)) and \( \mathcal{C}_{t_0} \) is simply connected. We claim that \( \tilde{\Phi}_{t_0} \) is surjective. Then \( \tilde{\Phi}_{t_0} \) is a homeomorphism. By Lemma 54 it is proved that for any \( r := (f, r_2, u_0) \in \mathcal{X}^J \) there exists a function \( q' \in \bigcup \mathcal{U}_p^J \) such that \( \Phi^J(q')|_{t=t_0} = r(t_0) \) where \( r^*_t = r^*(t) := (f^*(t), r^*_2(t), u^*_0(t)) \) for \( r^* := (f^*, r^*_2, u^*_0) \in \mathcal{X}^J \). Since \( \Phi^J \) is an open map, there exists a deformation \( q \in \bigcup \mathcal{U}_p^J \) of \( q' \) such that \( \Phi^J(q)|_{t=t_0} = r(t) \) on a neighborhood of \( t_0 \). More precisely there exists sufficiently small \( \epsilon > 0 \) such that any \( r^* \in \mathcal{X}^J \) with

\[
d(r^*, \Phi^J(q'))_{W_2^j \times (W_2^j)_{\text{div}} \times C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3} < \epsilon
\]

(distance) is in \( \text{Im} \Phi^J \). Let

\[
(C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3)_{\text{div}} := \{ v \mid v = \nabla \cdot w \ (\exists w \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3) \}.
\]

It is obtained that for small \( t_1 > 0 \),

\[
d(r(t), \Phi^J(q')|_{t})_{C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3 \times (C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3)_{\text{div}} \times C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3} < \epsilon \ (t_0 \leq t < t_0 + t_1)
\]

(distance). By formula (188) and formula (190) there exists \( q \in \bigcup \mathcal{U}_p^J \) such that \( \Phi^J(q)|_{t} = r(t) \ (t_0 \leq t < t_0 + t_1) \). So the image of \( \tilde{\Phi}_{t_0} \) contains the inductive limit \( \mathcal{X}^J \) for \( J \geq t_0 \). Observe that the completion of \( \mathcal{X}^J \) in \( \mathcal{X}^{(k),J} \) coincides with the latter space so that the completion of \( \mathcal{C}_{t_0}^\infty \) in \( \mathcal{C}_{t_0} \) does with \( \mathcal{C}_{t_0} \). From these the assertion is confirmed. Thus \( \tilde{\Phi}_{t_0} \) is a homeomorphism. \( \square \)

Remark 56. Navier-Stokes condition \( \mathcal{A} \) (see Remark 65) is satisfied.

We obtain a bijection \( \Phi : \mathcal{X} := \coprod_{t_0} \mathcal{X}_{t_0} \to \mathcal{C} := \coprod_{t_0} \mathcal{C}_{t_0} \) and introducing a sheaf structure to \( \mathcal{C} \) induced from \( \mathcal{X} \) through this bijection a sheaf isomorphism, where the topology of \( I \) is generated by \( J \)'s. We thus conclude as follows.

Corollary 57. \( \Phi : \mathcal{X} \to \mathcal{C} \) is a sheaf isomorphism.

Let \( (f, r_2, u_0) \in \mathcal{X}^{(k),J} \). Then there exists a section \( \tilde{p} := (u, \nabla p) \) of \( \mathcal{X} \) such that \( \Phi(\tilde{p}) = (f, r_2, u_0) \) on \( I_1 \), where \( I_1 := [0, T_1) \) \((0 < T_1 \leq T) \) (or \( I_1 := [0, T] \)) is maximal and \( \tilde{p} \) is locally unique in this topology of \( I_1 \) because \( \Phi \) is an isomorphism. Hence \( \tilde{p} \) is unique. Consider the following equation on \( I_1 \):

\[
\left[
\begin{array}{c}
\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u + \nabla p \\
\nabla \cdot u \nabla \\
u |_{t=0}
\end{array}
\right] := \Phi((u, \nabla p)) = \left[
\begin{array}{c}
f \\
r_2 \\
u_0
\end{array}
\right].
\]

(191)
We say \((u, \nabla p)\) satisfies equation (7) on \(I_1\) if it satisfies equation (191). Now we proved the following.

**Theorem 58.** Let \(\nu > 0\). Let \(k \in \mathbb{N}\). Let \(I := [0, T] (T > 0)\). Let

\[
(f, r_2, u_0) \in W^{k,k}_3 \times (W^{k,k+2}_3)_{\text{div}} \times H^{k+2}(\mathbb{R}^3/\mathbb{Z}^3)^3
\]

(192)
such that \(\nabla \cdot u_0 = r_2(0)\). Then there exists an unique \((u, \nabla p)\) satisfying equation (7) on \(I_1 := [0, T_1) (0 < T_1 \leq T)\) (or on \(I_1 := [0, T]\)) and that \(I_1\) is maximal.

We introduced to \(\mathcal{Y}\) the relative topology induced from (143) (which depends on \(k\)) and defined the Banach manifold \(\bigcup U_p^I\) in Corollary 53 and the sheaf \(\mathcal{Y}\) after the proof of Lemma 55. Now we remark the following:

**Remark 59.** In fact \((u, \nabla p) \in \mathcal{Y}(I_1)\).

We are going to prove Theorem 2. That \(\mathcal{Y} = \mathcal{X}\) is proved by Navier-Stokes conditions (see Remark 65). Since \((f, 0, u_0) \in A\) and \(\mathcal{Y} = \Phi(\mathcal{X})\) there exists \(p \in \mathcal{X}\) such that \(\Phi(p) = (f, 0, u_0)\), which defines a smooth solution of equation (7). We formalize this in the following way.

Let \(\Omega \subset \mathbb{R}^3/\mathbb{Z}^3\). Let

\[
W_{\Omega,n} := C^\infty(I, C^\infty(\Omega)^n),
\]

(193)

\[
(W_{\Omega,1})_\nabla := \{v \mid v = \nabla w (\exists w \in W_{\Omega,1})\},
\]

(194)

\[
(W_{\Omega,3})_{\text{div}} := \{v \mid v = \nabla \cdot w (\exists w \in W_{\Omega,3})\},
\]

(195)

\[
\mathcal{X}_\Omega := W_{\Omega,3} \times (W_{\Omega,1})_\nabla.
\]

(196)

**Lemma 60.** Let \((f, r_2, u_0) \in \mathcal{X}\). Let \(x \in \mathbb{R}^3/\mathbb{Z}^3\). Then there exists \((u^x, \nabla p^x) \in \mathcal{X}_{\{x\}}\) satisfying equation (7) on \(I \times \{x\}\).

**Proof.** The assertion easily follows. \(\square\)

**Lemma 61.** Let \((f, r_2, u_0) \in \mathcal{X}\). Let \(x \in \mathbb{R}^3/\mathbb{Z}^3\). Then there exist a compact neighbourhood \(K_x\) of \(x\) and \((u^{K_x}, \nabla p^{K_x}) \in \mathcal{X}_{K_x}\) satisfying equation (7) on \(I \times K_x\).

**Proof.** By Lemma 60 it follows that there exists \((u^x, \nabla p^x) \in \mathcal{X}_{\{x\}}\) satisfying equation (7) on \(I \times \{x\}\). Extend it arbitrarily to \(I \times (\mathbb{R}^3/\mathbb{Z}^3)\) to obtain \((u', \nabla p') \in \mathcal{X}\). Note that

\[
d((f, r_2, u_0), \Phi((u', \nabla p')))_{W_{\{x\},3} \times (W_{\{x\},1})_{\text{div}} \times C^\infty(\{x\})}^3
\]

(197)

(distance) is small for any \(z\) around \(x\). Then since by Lemma 30 \(\Phi : \mathcal{X} \rightarrow \mathcal{Y}\) is an open map there exist a compact neighbourhood \(K_x\) of \(x\) and \((u^{K_x}, \nabla p^{K_x}) \in \mathcal{X}_{K_x}\) satisfying equation (7) on \(I \times K_x\). The assertion follows. \(\square\)

**Remark 62.** Navier-Stokes condition 5 (see Remark 65) is satisfied.
Lemma 63. $\mathcal{Y} = \mathcal{X}$.

Proof. Let $(f, r_2, u_0) \in \mathcal{X}$. By Lemma 61 it is obtained that there exists a family $\{ (u^{K_x}, \nabla p^{K_x}) \}_x$ such that each $(u^{K_x}, \nabla p^{K_x})$ satisfies equation (7). Since $\mathbb{R}^3/\mathbb{Z}^3$ is compact we may obtain a finite set $\{ x_\lambda \}$ such that each $K_{x_\lambda}$ intersects with another in a set of Lebesgue measure 0 and $\bigcup_{\lambda} K_{x_\lambda} = \mathbb{R}^3/\mathbb{Z}^3$. It follows that there exists $(u, \nabla p) \in (L^2(I \times (\mathbb{R}^3/\mathbb{Z}^3))^3)^2$ that is smooth a.e. and satisfies equation (7) a.e. By Lemma 30 and Theorem 58 $\Phi : \mathcal{X} \to \mathcal{Y}$ is a homeomorphism. Extend $\Phi^{-1}$ to a continuous map from $\mathcal{Y} \subset \mathcal{X}$ to

$$\{(u, \nabla p) \in (L^2(I \times (\mathbb{R}^3/\mathbb{Z}^3))^3)^2 \mid u, \nabla p \text{ are smooth a.e.}\}.$$  

(198)

Since the set $\{ x_\lambda \}$ is finite, considering the convolutions $(u_\epsilon, \nabla p_\epsilon)$ with mollifiers it is proved that $(u, \nabla p) \in \Phi^{-1}(\mathcal{Y})$ and thus

$$(u, \nabla p) = \Phi^{-1}((f, r_2, u_0))$$

(199)

(note that $(u_\epsilon|_{K_{x_\lambda}}, \nabla p_\epsilon|_{K_{x_\lambda}})$ is convergent as elements in $\mathcal{X}_{K_{x_\lambda}}$). Let $(s, y) \in I \times (\mathbb{R}^3/\mathbb{Z}^3)$ be a singular point of $(u, \nabla p)$. Take another finite decomposition $K_y \cup \bigcup_{\mu} K_{x_\mu} = \mathbb{R}^3/\mathbb{Z}^3$ and construct $(u_1, \nabla p_1)$ that is smooth at $(s, y)$ and satisfies equation (7) a.e. Since

$$(u, \nabla p) = \Phi^{-1}((f, r_2, u_0)) = (u_1, \nabla p_1)$$

(200)

it follows that $(u, \nabla p)$ is smooth at $(s, y)$. This contradiction shows that $(u, \nabla p) \in \mathcal{X}$. Since

$$(f, r_2, u_0) = \Phi((u, \nabla p)) \in \mathcal{Y}$$

(201)

it is concluded that $\mathcal{X} \subset \mathcal{Y}$. Since the other inclusion is trivial the assertion follows. \qed

Remark 64. Navier-Stokes conditions (see Remark 65) are used.

Proof of Theorem 2. Since $(f, 0, u_0) \in \mathcal{X} = \mathcal{Y}$ (see Lemma 63) there exists $p \in \mathcal{X}$ such that $\Phi(p) = (f, 0, u_0)$, which defines a solution of (7). Further since $\Phi : \mathcal{X} \to \mathcal{Y}$ is a homeomorphism the solution is unique. The assertion follows. \qed

Remark 65. In the proof of Theorem 2 we used the following Navier-Stokes conditions (which are not assumptions) and conclude $\Phi$ is a homeomorphism.

1. $\Phi : \mathcal{X} \to \mathcal{Y}$ is a $C^\infty$-map such that $d\Phi(p) : T_p \mathcal{X} \to T_{\Phi(p)} \mathcal{Y}$ is a linear isomorphism.
2. Each seminorm $p_k$ of $\mathcal{Y}$, which is induced from the relative topology from
\[ W_3^{k,k} \times (W_3^{k,k+2})_{\text{div}} \times H^{k+2}(\mathbb{R}^3/\mathbb{Z}^3)^3, \]  

is actually a norm, that is, it satisfies the condition that $p_k(r) = 0$ implies $r = 0$.

3. The above holds if we replace $I$ with $J$.

4. For any $r \in \mathcal{X}$ there exists $q \in \bigcup_p U^J_p$ such that $\Phi^J(q)|_t = r(t) \ (t_0 \leq t < t_0 + t_1)$ for small $t_1$.

5. Let $(f, r_2, u_0) \in \mathcal{X}$. Let $x \in \mathbb{R}^3/\mathbb{Z}^3$. Then there exist a compact neighborhood $K_x$ of $x$ and $(u_{K_x}, \nabla p_{K_x}) \in \mathcal{X}_3$ satisfying equation (7) on $I \times K_x$.

Proof of Corollary 3. By Theorem 2 there exists an unique solution $(u, \nabla p)$ of equation (33) on $[0, T]$ for any $T > 0$ and patching the solutions there exists an unique global solution of the equation on $I$. The assertion follows.

\[ \square \]

8 Exact solutions

We obtain the exact solutions in the following way.

**Theorem 66.** Let $\nu > 0$. Let $I := [0, T] \ (T > 0)$. Let $u_0 \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3$ such that $\nabla \cdot u_0 = 0$ and let $f \in W_3$. Then the formal power series solution
\[ (u, \nabla p) = (\sum_m c_m t^m, \sum_m \nabla q_m t^m) \]  

of equation (7), where
\[ \begin{cases} 
    c_m = \sum_{L \in \mathbb{Z}^3} a_{L,m} e^{2\pi i L \cdot x} \quad (a_{L,m} \in \mathbb{C}^3), \\
    q_m = \sum_{L \in \mathbb{Z}^3} b_{L,m} e^{2\pi i L \cdot x} \quad (b_{L,m} \in \mathbb{C}) 
\end{cases} \]  

are formal Fourier series, is unique.

Proof. From $u|_{t=0} = u_0$, it is obtained that $c_0 = u_0$. From
\[ \nabla \cdot (\text{the first formula of equation (7)}) \]  

and the second formula of equation (7), $\Delta q_m$ (and thus $\nabla q_m$) is determined from $c_0, \ldots, c_m$. Then from the first formula of equation (7), $(m + 1)c_{m+1}$ is determined from $\nabla q_m, c_0, \ldots, c_m$. By induction all $c_m, \nabla q_m$ are determined from $u_0$. Thus the formal power series solution $(u, \nabla p)$ is unique. \[ \square \]
Remark 67. In the above \( q_m \) is not unique.

Let \( (u, \nabla p) \) be a smooth solution of (2). Expand \( u \) and \( p \) (not \( \nabla p \)) in the following way:

\[
(u, p) = \left( \sum_{m=0}^{\infty} \sum_{L \in \mathbb{Z}^3} \tilde{a}_{L,m} e^{2\pi i L \cdot x} e_m(t), \sum_{m=0}^{\infty} \sum_{L \in \mathbb{Z}^3} \tilde{b}_{L,m} e^{2\pi i L \cdot x} e_m(t) \right),
\]

where \( \tilde{a}_{L,m} \in \mathbb{C}^3 \), \( \tilde{b}_{L,m} \in \mathbb{C} \) and \( e_m(t) \) is the \( m \)-th orthogonal polynomial in \( L^2([0,T]) \). From this and Theorem 66 the smooth solution \((u, \nabla p)\) may be calculated.

9 Hodge Theory on \( \mathbb{R}^3 \)

Let \( k \in \mathbb{Z}_{\geq 0} \). Let \( H^k(\mathbb{R}^3) \) be the Sobolev space. Let

\[
H^\infty(\mathbb{R}^3) := \bigcap_k H^k(\mathbb{R}^3).
\]

(207)

Let \( \nabla \cdot (\cdot) : u \in H^\infty(\mathbb{R}^3)^3 \rightarrow \nabla \cdot u \in H^\infty(\mathbb{R}^3) \). Let

\[
(H^\infty(\mathbb{R}^3))_\nabla := \{ v \mid v = \nabla w \ (\exists w \in H^\infty(\mathbb{R}^3)) \}.
\]

(208)

The following is a special case of [3], Chapter VIII, Section 3, (3.2) Theorem.

Theorem 68. \( H^\infty(\mathbb{R}^3)^3 = \text{Ker}(\nabla \cdot (\cdot)) \oplus (H^\infty(\mathbb{R}^3))_\nabla \).

The following is obtained from [9], Chapter VI, Section 7, Theorem (Sobolev’s Lemma).

Theorem 69. Let \( K \subset \mathbb{R}^3 \) be a compact set. Let \( k, l \in \mathbb{Z}_{\geq 0} \) be such that \( \frac{3}{2} > \frac{3}{2} + k \). Let \( H^1(K) \) be the Sobolev space. Then there exists \( C > 0 \) such that

\[
||u||_{C^{k}(K)} \leq C||u||_{H^1(K)}.
\]

(209)

Let \( k \in \mathbb{Z}_{\geq 0} \). Write as

\[
\limsup_{|x| \rightarrow \infty} |\partial^j_x v(x)|^2
\]

(210)

the supremum of \( \limsup_{j \rightarrow \infty} |\partial^j_x v(x_j)|^2 \), where \( \{x_j\} \) runs all the sequences such that \( |x_j| \rightarrow \infty \) as \( j \rightarrow \infty \). Let

\[
\Gamma^k_0(\mathbb{R}^3) := \{ v \in H^k(\mathbb{R}^3)^3 \mid ||v||^2 := ||v||^2_{H^k(\mathbb{R}^3)} + \sum_{|\alpha| \leq k} \limsup_{|x| \rightarrow \infty} |\partial^\alpha_x v(x)|^2 < \infty \}.
\]

(211)
Let
\[
\Gamma^k(\mathbb{R}^3) := \{ v \in \Gamma_0^k(\mathbb{R}^3) \mid \sum_{|\alpha| \leq k} \limsup_{|x| \to \infty} |\partial_\alpha^w v(x)|^2 = 0 \},
\]

and
\[
\Gamma^\infty(\mathbb{R}^3) := \bigcap_k \Gamma^k(\mathbb{R}^3).
\]

**Lemma 70.** \(H^\infty(\mathbb{R}^3) = \Gamma^\infty(\mathbb{R}^3)\).

**Proof.** Let \( w \in H^\infty(\mathbb{R}^3) \). Let \( K \subset \mathbb{R}^3 \) be a compact neighbourhood of 0. Let \( w'(y) := w(x + y) \ (y \in K) \). By Theorem 69 it is obtained that for \( l > \frac{3}{2} + k \) there exists \( C > 0 \) such that
\[
||w'||_{C^k(K)} \leq C||w'||_{H^l(K)},
\]
so that since \( w \in H^\infty(\mathbb{R}^3) \),
\[
||w(x + \cdot)||_{C^k(K)} \leq C||w(x + \cdot)||_{H^l(K)} \to 0 \ (|x| \to \infty).
\]
Thus
\[
\sum_{|\alpha| \leq k} \limsup_{|x| \to \infty} |\partial_\alpha^w w(x)|^2 = 0.
\]
It follows that \( w \in \Gamma^\infty(\mathbb{R}^3) \) and \( H^\infty(\mathbb{R}^3) \subset \Gamma^\infty(\mathbb{R}^3) \). Since the other inclusion is trivial the assertion follows. \( \square \)

Let
\[
(\Gamma^\infty(\mathbb{R}^3))_\nabla := \{ v \mid v = \nabla w \ (\exists w \in \Gamma^\infty(\mathbb{R}^3)) \}.
\]

**Corollary 71.** \( \Gamma^\infty(\mathbb{R}^3)_\nabla = (\text{Ker}(\nabla \cdot \cdot)) \cap \Gamma^\infty(\mathbb{R}^3)^3 \oplus (\Gamma^\infty(\mathbb{R}^3))_\nabla \).

**Proof.** By Lemma 70 \( H^\infty(\mathbb{R}^3) = \Gamma^\infty(\mathbb{R}^3) \) so that \( \Gamma^\infty(\mathbb{R}^3)^3 = H^\infty(\mathbb{R}^3)^3 \) and \( (\text{Ker}(\nabla \cdot \cdot)) \cap \Gamma^\infty(\mathbb{R}^3)^3 \oplus (\Gamma^\infty(\mathbb{R}^3))_\nabla = \text{Ker}(\nabla \cdot \cdot) \oplus (H^\infty(\mathbb{R}^3))_\nabla \). By Theorem 68 the assertion follows. \( \square \)

**10 Linear evolution equations on \( \mathbb{R}^3 \)**

Let \( \mathcal{P} : \Gamma^\infty(\mathbb{R}^3)^3 \to \text{Ker}(\nabla \cdot \cdot) \) be the projection. For \( n \in \mathbb{N} \) let
\[
W_n := C^\infty(I, \Gamma^\infty(\mathbb{R}^3)^n).
\]
We shall prove the existence and uniqueness of a solution of the equation
\[
\begin{cases}
\dot{h} + \mathcal{P}((u \cdot \nabla) \mathcal{P} h + (\mathcal{P} h \cdot \nabla) u - \nu \Delta \mathcal{P} h) - g = 0, \\
h(0) \in \Gamma^\infty(\mathbb{R}^3)^3,
\end{cases}
\]
for \( g \in C^\infty(I, \Gamma^\infty(\mathbb{R}^3)^3) \).

Let \( k \in \mathbb{Z}_{>0} \). Let \( I := [0, T] \) and \( u \in C^\infty(I, \Gamma^\infty(\mathbb{R}^3)^3) \). Define a linear operator \( A'_k(t) \) on \( \Gamma^k(\mathbb{R}^3)^3 \) by

\[
-A'_k(t)h := \mathcal{P}((u \cdot \nabla)h + (\mathcal{P}h \cdot \nabla)u - \nu \Delta \mathcal{P}h)
\]

for \( h \in \Gamma^\infty(\mathbb{R}^3)^3 \). Since the adjoint \((A'_k(t))^*\) of \( A'_k(t) \) is densely defined \( A'_k(t) \) is closable. Let \( A_k(t) \) be the closure of \( A'_k(t) \). For \( k_1, k_2 \in \mathbb{Z}_{>0} \cup \{ \infty \} \) let

\[
W^{k_1,k_2} := C^{k_1}(I, \Gamma^{k_2}(\mathbb{R}^3)^n).
\]

**Lemma 72.** Let \( I := [0, T] \) (\( T > 0 \)). Let \( \nu > 0 \). Assume \( u \in W_3 \). Then the equation

\[
\dot{h} - A_k(t)h = 0
\]

has at most one solution \( h \in W^{1,k}_3 \) for any initial condition \( h(0) \in \Gamma^\infty(\mathbb{R}^3)^3 \).

**Proof.** Let \( h \in \Gamma^\infty(\mathbb{R}^3)^3 \). Observe that

\[
-A_k(t)h = \mathcal{P}(\nu(\nabla - A(t))^2)\mathcal{P}h + \mathcal{P}\mathcal{B}(t)\mathcal{P}h
\]

for some \( W_1 \)-coefficiential \( 3 \times 1 \) and \( 3 \times 3 \) matrices \( A(t), \mathcal{B}(t) \). Thus by an elementary argument

\[
\text{Re} < -A_k(t)h, h >_{\Gamma^k(\mathbb{R}^3)^3} \geq -c < h, h >_{\Gamma^k(\mathbb{R}^3)^3}
\]

for some \( c > 0 \). Taking limit formula (224) also holds for \( h \in D(-A_k(t)) \), where \( D(-A_k(t)) \) is the domain of \(-A_k(t)\).

Let \( h \) be the solution of equation (223) and \( H = e^{-\nu t}h \). Observe that \( h \) depends on \( t \). Then

\[
< \dot{H}, H >_{\Gamma^k(\mathbb{R}^3)^3} = < (A_k(t) - c)H, H >_{\Gamma^k(\mathbb{R}^3)^3}.
\]

Assume \( h(0) = 0 \). Let \( t_0 \in [0, T] \). By formula (224)

\[
\|H(t_0)\|_{\Gamma^k(\mathbb{R}^3)^3}^2 - \|H(0)\|_{\Gamma^k(\mathbb{R}^3)^3}^2 = 2 \int_0^{t_0} \text{Re} < (A_k(t) - c)H, H >_{\Gamma^k(\mathbb{R}^3)^3} \, dt
\]

\leq 0.

It follows that

\[
\|H(t_0)\|_{\Gamma^k(\mathbb{R}^3)^3}^2 \leq \|H(0)\|_{\Gamma^k(\mathbb{R}^3)^3}^2 = 0.
\]

Hence \( h(t_0) = 0 \). Since \( t_0 \in [0, T] \) is arbitrary \( h = 0 \). Assume \( h_1, h_2 \) are two solutions. Then \( h_2(0) - h_1(0) = 0 \) and \( h_2 - h_1 \) is a solution of the equation (223). Thus the above argument shows that \( h_2 - h_1 = 0 \). From this the assertion follows.

**Lemma 73.** \( A_k(t) \) generates a \( C^0 \)-semigroup for each \( t \in I \).
Proof. Take \( c \) as in Lemma 72. Since by definition the domain of \( A_k(t) \) is dense, so is that of \( (A_k(t) - c) \). Observe that

\[
\text{Re} < (A_k(t) - c)h, h >_{\Gamma^k(\mathbb{R}^3)^3} \leq 0
\]  

(230)

for all \( h \in D(A_k(t) - c) \), where \( D(A_k(t) - c) \) is the domain of \( (A_k(t) - c) \). In particular the image of \( \text{Id} - (A_k(t) - c) \) is closed. The adjoint operator \((\text{Id} - (A_k(t) - c))^*\) of \( (A_k(t) - c) \) is clearly injective. Hence the image of \( \text{Id} - (A_k(t) - c) \) is the whole space \( \Gamma^k(\mathbb{R}^3)^3 \). By Theorem 18, \( (A_k(t) - c) \) generates a contraction \( C^0 \)-semigroup. The assertion follows. \( \square \)

Lemma 74. Let \( k, k' \in \mathbb{Z}_{\geq 0} \). Let \( \{e^{sA_k(t)}\}_{s \geq 0} \) be a \( C^0 \)-semigroup on \( \Gamma^k(\mathbb{R}^3)^3 \) generated by \( A_k(t) \) for each \( t \). Then \( e^{sA_k(t)}h_0 = e^{sA_{k'}(t)}h_0 \) for any \( h_0 \in \Gamma^k(\mathbb{R}^3)^3 \) and \( s \in I := [0, T] \).

Proof. Assume without loss of generality \( k \leq k' \). Observe that by assumption \( h_0 \in \Gamma^k(\mathbb{R}^3)^3 \) so that \( h_1(s, x) := e^{sA_k(t)}h_0 \in W^{1,k}_3 \) and \( h_2(s, x) := e^{sA_{k'}(t)}h_0 \in W^{1,k'}_3 \). Then since \( -A_k(t)|_{\Gamma^{k'}(\mathbb{R}^3)^3} = -A_{k'}(t) \) it follows that \( h_1 \) and \( h_2 \) are solutions of

\[
\begin{cases}
\frac{\partial h}{\partial x} + A_k(t)h = 0, \\
h(0) = h_0. 
\end{cases}
\]

(231)

Since \( A_k(t) \) is dissipative an elementary argument shows that the solution \( h \in W^{1,k}_3 \) of this equation is unique. Thus \( e^{sA_k(t)}h_0 = h_1(s, x) = h_2(s, x) = e^{sA_{k'}(t)}h_0 \) for \( s \in I \). The assertion follows. \( \square \)

Let \( B(\Gamma^k(\mathbb{R}^3)^3) \) be the set of continuous linear operators on \( \Gamma^k(\mathbb{R}^3)^3 \).

Theorem 75. Assume

\[
\| \prod_{j=1}^{J} (\lambda - A_k(t_j))^{-1} \|_{B(\Gamma^k(\mathbb{R}^3)^3)} \leq M_k(\lambda - \beta_k)^{J} \quad (\lambda > \beta_k).
\]

(232)

for \( 0 \leq t_1 \leq \cdots \leq t_J \leq T \) \((J = 1, 2, \ldots)\), where the product \( \prod \) is time-ordered, i.e. a factor with larger \( t_j \) stands to the left of ones with smaller \( t_j \). Then there exists an operator-valued function \( U(t, s) \) \((0 \leq s \leq t \leq T)\) on \( \Gamma^\infty(\mathbb{R}^3)^3 \) that satisfies the following.

(a) \((s, t) \mapsto U(t, s)h_0 \) \((h_0 \in \Gamma^\infty(\mathbb{R}^3)^3) \) is continuous, \( U(s, s) = \text{Id} \),

\[
\|U(t, s)\|_{B(\Gamma^k(\mathbb{R}^3)^3)} \leq M_k e^{\beta_k(t-s)}.
\]

(233)

(b) If \( s \leq r \leq t \) then \( U(t, s) = U(t, r)U(r, s) \).

(c) For \( h_0 \in \Gamma^\infty(\mathbb{R}^3)^3 \) and \( s \in [0, T) \),

\[
D_t^+ U(t, s)h_0|_{t=s} = A_0(s)h_0,
\]

(234)
where \( D^+ \) denotes the right derivative.

(d) For \( h_0 \in \Gamma^\infty(\mathbb{R}^3)^3 \) and \( 0 \leq s \leq t \leq T \),
\[
\frac{\partial}{\partial s} U(t, s) h_0 = -U(t, s) A_0(s) h_0. 
\tag{235}
\]

(e) For \( h_0 \in \Gamma^\infty(\mathbb{R}^3)^3 \) and \( 0 \leq s \leq t \leq T \),
\[
\frac{\partial}{\partial t} U(t, s) h_0 = A_0(t) U(t, s) h_0. \tag{236}
\]

Proof. By Lemma 73, \( A_k(t) \) generates a \( C^0 \)-semigroup \( \{ e^{sA_k(t)} \}_{s \geq 0} \) and since \( \Gamma^\infty(\mathbb{R}^3)^3 \) is dense in \( \Gamma^k(\mathbb{R}^3)^3 \), it is obtained by Lemma 74 that
\[
e^{sA_k(t)}|_{\Gamma^k(\mathbb{R}^3)^3} = e^{sA_k(t)}. \tag{237}
\]

By Lemma 8,
\[
\left\| \prod_{j=1}^J e^{s_j A_0(t_j)} \right\|_{\Gamma^k(\mathbb{R}^3)^3} \leq M_k e^{\sum_{j=1}^J s_j}, \tag{239}
\]
for \( 0 \leq t_1 \leq \cdots \leq t_J \leq T \) and \( s_j \geq 0 \). The product \( \prod \) is time-ordered, i.e. a factor with larger \( t_j \) stands to the left of ones with smaller \( t_j \).

Observe that \( A_0(t)|_{\Gamma^k(\mathbb{R}^3)^3} = A_k(t) \). Let \( L(\Gamma^{k+2}(\mathbb{R}^3)^3, \Gamma^k(\mathbb{R}^3)^3) \) be the set of continuous linear operators from \( \Gamma^{k+2}(\mathbb{R}^3)^3 \) to \( \Gamma^k(\mathbb{R}^3)^3 \). Let \( A_{0,n}(t) = A_0(\lceil nt/T \rceil/n) \), where \( \lceil \rceil \) denotes the Gauss symbol. Then
\[
\| A_{0,n}(t) - A_{0}(t) \|_{L(\Gamma^{k+2}(\mathbb{R}^3)^3, \Gamma^k(\mathbb{R}^3)^3)} \to 0 \ (n \to \infty). \tag{241}
\]

Let
\[
U_n(t, s) = e^{(t-s)A_0(k'T/n)}, \quad (k'T/n \leq s \leq t \leq (k' + 1)T/n), 
\tag{242}
\]
\[
U_n(t, s) = e^{(t-l'T/n)A_0(l'T/n)} e^{(T/n)A_0((l'-1)T/n)} \cdots e^{(T/n)A_0((k'+1)T/n)} e^{((k'+1)T/n-s)A_0(k'T/n)}, 
\tag{243}
\]
\[
(k'T/n \leq s < (k' + 1)T/n, l'T/n \leq t < (l' + 1)T/n, k' < l'). \tag{244}
\]

By inequality (238)-(240),
\[
\| U_n(t, s) \|_{B(\Gamma^k(\mathbb{R}^3)^3)} \leq M_k e^{\beta_k (t-s)}. \tag{246}
\]
Thus $U_n(t, s)$ satisfies (a), (b). For $h_0 \in \Gamma^\infty(\mathbb{R}^3)^3$ and $t \neq k''T/n$ ($k'' = 0, 1, \ldots, n$),

$$\frac{\partial}{\partial t} U_n(t, s)h_0 = A_{0,n}(t)U_n(t, s)h_0,$$

and for $h_0 \in \Gamma^\infty(\mathbb{R}^3)^3$ and $s \neq l''T/n$ ($l'' = 0, 1, \ldots, n$),

$$\frac{\partial}{\partial s} U_n(t, s)h_0 = -U_n(t, s)A_{0,n}(s)h_0.$$  \hspace{1cm} (247)

Let $h_0 \in \Gamma^\infty(\mathbb{R}^3)^3$. Then by inequality (246),

$$\|U_n(t, s)h_0 - U_m(t, s)h_0\|_{\Gamma^k(\mathbb{R}^3)^3} \leq M_k \|A_{0,n}(r) - A_{0,m}(r)\|_{\Gamma(\mathbb{R}^3)^3} \leq M_k M_k + e^{\max \{\beta_k, \beta_k+1\}(t-s)} \|h_0\|_{\Gamma^{k+2}(\mathbb{R}^3)^3} \times \int_s^t \|A_{0,n}(r) - A_{0,m}(r)\|_{L(H^{k+2}(\mathbb{R}^3)^3, \Gamma^k(\mathbb{R}^3)^3)} dr.$$  \hspace{1cm} (254)

$k \in \mathbb{Z}_>0$ is arbitrary and thus for any $h_0 \in \Gamma^\infty(\mathbb{R}^3)^3$ and $0 \leq s \leq t \leq T$,

$$U(t, s)h_0 := \lim_{n \to 0} U_n(t, s)h_0$$  \hspace{1cm} (255)

exists. Note that $(s, t) \to U(t, s)$ is continuous because the convergence is uniform in $0 \leq s \leq t \leq T$. Since $U_n(t, s)$ satisfies (a), (b) so does $U(t, s)$. For $h_0 \in \Gamma^\infty(\mathbb{R}^3)^3$,

$$\|U_n(t, s)h_0 - e^{(t-s)A_{0}(s)}h_0\|_{\Gamma^k(\mathbb{R}^3)^3}$$  \hspace{1cm} (256)

and for $h_0 \in \Gamma^\infty(\mathbb{R}^3)^3$ and $s \neq l''T/n$ ($l'' = 0, 1, \ldots, n$),

$$\|U_n(t, s)h_0 - e^{(t-s)A_{0}(s)}h_0\|_{\Gamma^k(\mathbb{R}^3)^3} \leq M_k M_k + e^{\max \{\beta_k, \beta_k+1\}(t-s)} \|h_0\|_{\Gamma^{k+2}(\mathbb{R}^3)^3} \times \int_s^t \|A_{0,n}(r) - A_{0,r}(r)\|_{L(H^{k+2}(\mathbb{R}^3)^3, \Gamma^k(\mathbb{R}^3)^3)} dr.$$  \hspace{1cm} (259)

It follows that for $h_0 \in \Gamma^\infty(\mathbb{R}^3)^3$,

$$\|U(t, s)h_0 - e^{(t-s)A_{0}(s)}h_0\|_{\Gamma^k(\mathbb{R}^3)^3}$$  \hspace{1cm} (261)

$$\leq M_k M_k + e^{\max \{\beta_k, \beta_k+1\}(t-s)} \|h_0\|_{\Gamma^{k+2}(\mathbb{R}^3)^3} \times \int_s^t \|A_{0,n}(r) - A_{0,r}(r)\|_{L(H^{k+2}(\mathbb{R}^3)^3, \Gamma^k(\mathbb{R}^3)^3)} dr.$$  \hspace{1cm} (263)
Since $k \in \mathbb{Z}_{\geq 0}$ is arbitrary this proves (c). Similarly it is shown that for $h_0 \in \Gamma^\infty(\mathbb{R}^3)^3, \quad
\frac{1}{\varepsilon}(U(t, s + \varepsilon) h_0 - U(t, s) h_0) = \frac{1}{\varepsilon}(h_0 - U(s, s) h_0) \quad (264)
\rightarrow -U(t, s) A_0(s) h_0 \quad (\varepsilon \rightarrow +0).

For $h_0 \in \Gamma^\infty(\mathbb{R}^3)^3, s \leq t$ it is obtained by formula (264) that

\begin{align*}
\frac{1}{\varepsilon}(U(t, s) h_0 - U(t, s - \varepsilon) h_0) &= U(t, s) \frac{1}{\varepsilon}(h_0 - U(s, s - \varepsilon) h_0) \quad (265)
\rightarrow -U(t, s) A_0(s) h_0 \quad (\varepsilon \rightarrow +0). \quad (266)
\end{align*}

They prove (d). (e) follows from (c) and (d).

**Lemma 76.** Let $I := [0, T]$ ($T > 0$). Let $\nu > 0$. Assume $u, g \in W_3$. Then the equation

\begin{equation}
\dot{h} - A_0(t) h - g = 0 \quad (271)
\end{equation}

has a solution $h \in W_3^{1, \infty}$ that is uniquely determined from $h(0) \in \Gamma^\infty(\mathbb{R}^3)^3$.

**Proof.** By Theorem 75 there exists $U(t, s)$ satisfying (a)-(e). Let

\begin{equation}
h := U(t, 0) h(0) + \int_0^t U(t, s) g(s) ds. \quad (272)
\end{equation}

Then $h \in W_3^{1, \infty}$ and it is a solution of equation (271). Let $h_1, h_2$ be two solutions of equation (271) then $h_2 - h_1$ is a solution of

\begin{equation}
\begin{cases}
\dot{h} - A_0(t) h = 0, \\
h|_{t=0} = 0.
\end{cases} \quad (273)
\end{equation}

By Lemma 72 this equation has the unique solution 0. Thus $h_2 - h_1 = 0$ and equation (271) has a solution $h \in W_3^{1, \infty}$ that is uniquely determined from $h(0) \in \Gamma^\infty(\mathbb{R}^3)^3$. The assertion follows. \qed

**Theorem 77.** Let $I := [0, T]$ ($T > 0$). Let $\nu > 0$. Assume $u, g \in W_3$. Then the equation

\begin{equation}
\dot{h} + \mathcal{P}(u \cdot \nabla) \mathcal{P} h + (\mathcal{P} h \cdot \nabla) u - \nu \Delta \mathcal{P} h - g = 0 \quad (274)
\end{equation}

has a solution $h \in W_3$ that is uniquely determined from $h(0) \in \Gamma^\infty(\mathbb{R}^3)^3$. 

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Proof. By Lemma 76, equation (274) has an unique solution $h$ in $W_3^{1,\infty}$. Since $h$ satisfies equation (274), if $\dot{h} \in W_3^{l,\infty}$ for $l \in \mathbb{N}$, it follows that $\dot{h} \in W_3^{l,\infty}$, that is, $h \in W_3^{l+1,\infty}$. Hence by induction it is proved that $h \in W_3$. The assertion follows. \end{proof}

11 Navier-Stokes equations on $\mathbb{R}^3$

Let $\nu > 0$ and Let $I := [0, T]$ ($T > 0$). Let

$$\tag{275} (W_1)^\nu := \{ v \mid v = \nabla w \ (\exists w \in W_1) \}.$$ 

We prove the following theorem.

**Theorem 78.** Let $\nu > 0$. Let $I := [0, T]$ ($T > 0$). Let $u_0 \in \Gamma^\infty(\mathbb{R}^3)^3$ such that $\nabla \cdot u_0 = 0$ and let $f \in W_3$. Then there exists an unique $(u, \nabla p) \in W_3 \times (W_1)^\nu$ such that

$$\begin{cases}
\frac{\partial u}{\partial t} = -(u \cdot \nabla) u + \nu \Delta u - \nabla p + f, \\
\nabla \cdot u = 0, \\
|u|_{t=0} = u_0, \\
\lim\sup\sup_{|x| \to \infty} \partial_x^2 u(t, x) = 0 (\forall t, \alpha). 
\end{cases} \tag{276}$$

The proof of Theorem 78 is essentially the same as that of Theorem 2. We describe the details.

Let

$$\Phi : W_3 \times (W_1)^\nu \to W_3 \times (W_3)_{\nabla} \times \Gamma^\infty(\mathbb{R}^3)^3 \tag{277}$$

be given by

$$\Phi(u, P) \mapsto \begin{bmatrix}
\dot{u} + (u \cdot \nabla) u - \nu \Delta u + P \\
\nabla \cdot u \\
u(t, x) 
\end{bmatrix}, \tag{278}$$

where

$$(W_3)_{\nabla} := \{ v \mid v = \nabla w \ (\exists w \in W_3) \}. \tag{279}$$

We introduce to $(W_1)^\nu$ the relative topology as a subset of $W_3$ and to $(W_3)_{\nabla}$ the relative topology as a subset of $W_1$. Then $\Phi$ is $C^\infty$ in $(u, P)$ in the sense of Fréchet derivatives (see Definition 31). We begin with proving the following lemma:

**Lemma 79.** The map

$$d\Phi((u, P)) : (h, \beta) \mapsto \begin{bmatrix}
\dot{h} + (u \cdot \nabla) h + (\beta \cdot \nabla) u - \nu \Delta h + \beta \\
\nabla \cdot h \\
h(t, x)
\end{bmatrix} \tag{280}$$

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is a linear isomorphism from

\[ W_3 \times (W_1)_\nabla \]  

(281)

to

\[ \{(a, b, c) \in W_3 \times (W_3)_{\text{div}} \times \Gamma^\infty(\mathbb{R}^3)^3 \mid \nabla \cdot c = b(0)\} \]  

(282)

Proof. By calculation \(d\Phi\) is given by the map (280). Let

\[ (a, b, c) \in \{(a, b, c) \in W_3 \times (W_3)_{\text{div}} \times \Gamma^\infty(\mathbb{R}^3)^3 \mid \nabla \cdot c = b(0)\} \]  

(283)

Let

\[ \begin{align*}
\dot{h} + (u \cdot \nabla)h + (h \cdot \nabla)u - \nu \Delta h + \beta &= a, \\
\nabla \cdot h &= b, \\
\h(0) &= c.
\end{align*} \]

(284, 285, 286)

From equation (285), \(h \in W_3\) is determined up to \(\text{Ker}(\nabla \cdot \cdot)\). Since \(\beta \in (W_1)_\nabla\), by Corollary 71, Theorem 77 and an elementary argument \(h \in W_3\) is uniquely determined from equation (284) and equation (286). Then from equation (284), \(\beta \in (W_1)_\nabla\) is uniquely determined. From this the assertion follows. \(\Box\)

Let

\[ \mathcal{X} := W_3 \times (W_1)_\nabla \]  

(287)

and

\[ \mathcal{X} := \{r := (f, r_2, u_0) \in W_3 \times (W_3)_{\text{div}} \times \Gamma^\infty(\mathbb{R}^3)^3 \mid \nabla \cdot u_0 = r_2(0)\} \]  

(288)

Then \(\Phi\) is a map from \(\mathcal{X}\) to \(\mathcal{X}\). Let

\[ \mathcal{Y} := \Phi(\mathcal{X}). \]  

(289)

Let \(p = (u, P) \in \mathcal{X}\) be an arbitrary point.

**Lemma 80.** \(d\Phi(p)\) is a topological isomorphism.

Proof. Note that \(\Phi\) is a \(C^1\)-map from \(\mathcal{X}\) to \(\mathcal{X}\) and from Lemma 79

\[ d\Phi(p) : (h, \beta) \mapsto \begin{bmatrix} \dot{h} + (u \cdot \nabla)h + (h \cdot \nabla)u - \nu \Delta h + \beta \\ \nabla \cdot h \\ h(0) \end{bmatrix} \]  

(290)

is a linear isomorphism and by the open mapping principle a topological linear isomorphism from

\[ T_p \mathcal{X} = \{(h, \beta) \in W_3 \times (W_1)_\nabla\} \]  

(291)

to

\[ T_{\Phi(p)} \mathcal{X} = \{(a, b, c) \in W_3 \times (W_3)_{\text{div}} \times \Gamma^\infty(\mathbb{R}^3)^3 \mid \nabla \cdot c = b(0)\} \]  

(292)

The assertion follows. \(\Box\)
Lemma 81. \( \mathcal{Y} \) is a Fréchet manifold.

Proof. By Lemma 80 \( d\Phi(p) \) is a topological isomorphism so that, since \( p \) is arbitrary, by Lemma 30 \( \Phi : \mathcal{X} \rightarrow \mathcal{Y} \) is an open map. Since \( \mathcal{Y} = \Phi(\mathcal{X}) \) it is a Fréchet manifold. The assertion follows. \( \square \)

Let
\[
\varphi_p : \mathcal{X} \xrightarrow{\sim} T_p \mathcal{X}
\]
and
\[
\psi_{\Phi(p)} : \mathcal{Y} \xrightarrow{\sim} T_{\Phi(p)} \mathcal{Y}
\]
be canonical isomorphisms. Let
\[
\mathcal{U}_p := \varphi_p^{-1}(T_p \mathcal{X}) (= \mathcal{X})
\]
and
\[
\mathcal{V}_{\Phi(p)} := \psi_{\Phi(p)}^{-1}(T_{\Phi(p)} \mathcal{Y}).
\]
Observe that by Lemma 80
\[
d\Phi(p) : T_p \mathcal{X} \xrightarrow{\sim} T_{\Phi(p)} \mathcal{Y}
\]
is a topological linear isomorphism. Let \( k \in \mathbb{N} \). Let
\[
(W^{k,k+2}_3)_{\text{div}} := \{ v \mid v = \nabla \cdot w \ (\exists w \in W^{k,k+2}_3) \}.
\]
Then introduce to \((W^{k,k+2}_3)_{\text{div}}\) the relative topology as a subset of \( W^{k,k+1}_1 \) and to \( \mathcal{Y} \) the relative topology as a subset of
\[
W^{k,k}_3 \times (W^{k,k+2}_3)_{\text{div}} \times \Gamma^{k+2}(\mathbb{R}^3)^3,
\]
which defines a seminorm \( p_k \) (it is in fact a norm) of \( \mathcal{Y} \). Let the system of open sets of the topology of \( \mathcal{Y} \) be
\[
\{ \mathcal{O} \}.
\]
Introduce to \( T_{\Phi(p)} \mathcal{Y} \) the topology induced from \( \theta_{\Phi(p)} := (\psi_{\Phi(p)})^{-1}|_{T_{\Phi(p)} \mathcal{Y}} \), i.e.
\[
\{ (\theta_{\Phi(p)}^{-1}(\mathcal{O})) \},
\]
to \( T_p \mathcal{X} \) the topology induced from \( \theta_{\Phi(p)} \circ d\Phi(p) \), i.e.
\[
\{ (d\Phi(p))^{-1} \circ \theta_{\Phi(p)}^{-1}(\mathcal{O}) \},
\]
to \( \mathcal{U}_p \) the topology induced from \( \theta_{\Phi(p)} \circ d\Phi(p) \circ \varphi_p|_{\mathcal{U}_p} \), i.e.
\[
\{ (\varphi_p|_{\mathcal{U}_p})^{-1} \circ (d\Phi(p))^{-1} \circ \theta_{\Phi(p)}^{-1}(\mathcal{O}) \},
\]
and to \( \mathcal{V}_{\Phi(p)} \) the topology induced from \( \theta_{\Phi(p)} \circ \psi_{\Phi(p)}|_{\mathcal{V}_{\Phi(p)}} \), i.e.
\[
\{ (\psi_{\Phi(p)}|_{\mathcal{V}_{\Phi(p)}})^{-1} \circ \theta_{\Phi(p)}^{-1}(\mathcal{O}) \}.
\]
Lemma 82. $U_p = \mathcal{X}$ and $\mathcal{V}_\Phi(p) = \mathcal{X}$.

Proof. From the definitions, using the canonical isomorphisms (293) and (294), the assertion follows. 

\[\square\]

Remark 83. $\Phi$ satisfies Navier-Stokes condition 1. That $p_k$ is a norm is Navier-Stokes condition 2. (See Remark 108.)

By Lemma 82 $U_p = \mathcal{X}$ and $\mathcal{V}_\Phi(p) = \mathcal{X}$. The topologies are induced by a seminorm $p_k$ such that $p_k(r)$ implies $r = 0$ (for $r \in \mathcal{Y}$). $q \mapsto d\Phi(q)$ is continuous with respect to the ordinary topology of $U_p$ and the topology of $T_{\Phi(p)}\mathcal{V}_\Phi(p)$ induced by $p_k$.

Lemma 84. Let $\mathcal{X}_p$ be a sufficiently small convex neighbourhood of $p \in \mathcal{X}$ identified with the corresponding neighbourhood of $0 \in T_q\mathcal{X}_p$ ($q \in \mathcal{X}_p$). Let $\{q_j\} \subset \mathcal{X}_p$ be a sequence. Then there exists a subsequence of $\{q_j\}$ convergent to $0 \in \mathcal{X}_p$ with respect to the ordinary topology.

Proof. Let $k_1, k_2 \in \mathbb{Z}_{\geq 0}$. Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^3$ and let

\[\Gamma^{k_2}(\Omega) := \{h|\Omega \mid h \in \Gamma^{k_2}(\mathbb{R}^3)\},\]

\[W^{k_1,k_2}_{\Omega,n} := C^{k_1}(I, \Gamma^{k_2}(\Omega)^n),\]

\[(W^{k_1,k_2+1}_{\Omega,1})^\odot := \{v \mid v = \nabla w \ (\exists w \in W^{k_1,k_2+1}_{\Omega,1})\}.\]

Let $(u_j, P_j) := q_j$. $K \subset \mathbb{R}^3$ be a compact set. By Theorem 41 possibly passing to a subsequence $(\partial_t^{k_1} u_j(t)|_K, \partial_t^{k_1} P_j(t)|_K)$ is convergent for fixed $t \in I$. Let

\[\lim_{j \to \infty} (\partial_t^{k_1} u_j(t)|_K, \partial_t^{k_1} P_j(t)|_K) := (\partial_t^{k_1} u(t)|_K, \partial_t^{k_1} P(t)|_K)\]

(308)

By diagonal argument the same formula holds for any $t_0 \in I'$, where $I'$ is a dense countable subset of $I$. By assumption $||(\partial_t^{k_1+1} \partial_x^{k_2} u_j, \partial_t^{k_1+1} \partial_x^{k_2} P_j)||_{W^{0,0}_{K_3} \times (W^{0,1}_{K_1})^\odot}$ is bounded on $I$ so that $(\partial_t^{k_1+1} \partial_x^{k_2} u(t_0), \partial_t^{k_1+1} \partial_x^{k_2} P(t_0))$ ($t_0 \in I'$) extends to a continuous function on $t \in I$. The resulting function is denoted by the same symbol. Since $||(\partial_t^{k_1+1} \partial_x^{k_2} u_j, \partial_t^{k_1+1} \partial_x^{k_2} P_j)||_{W^{0,0}_{K_3} \times (W^{0,1}_{K_1})^\odot}$ is bounded there exists $M > 0$ such that

\[||(\partial_t^{k_1} u_j(t)|_K, \partial_t^{k_1} P_j(t)|_K) - (\partial_t^{k_1} u(t_0)|_K, \partial_t^{k_1} P(t_0)|_K)||_{W^{0,k_2}_{K,3} \times (W^{0,k_2+1}_{K,1})^\odot} \leq M|t - t_0|\]

(309)

and

\[||(\partial_t^{k_1} u(t)|_K, \partial_t^{k_1} P(t)|_K) - (\partial_t^{k_1} u(t_0)|_K, \partial_t^{k_1} P(t_0)|_K)||_{W^{0,k_2}_{K,3} \times (W^{0,k_2+1}_{K,1})^\odot} \leq M|t - t_0|\]

(310)
For any $t \in I$ and any $\epsilon > 0$ there exists $t_0 \in A$ such that $M|t - t_0| < \frac{\epsilon}{3}$ so that by above for large $j$

\[
|||\langle \partial_t^{k_1} u_j(t) | K, \partial_t^{k_1} P_j(t) | |K \rangle - \langle \partial_t^{k_1} u(t) | K, \partial_t^{k_1} P(t) | |K \rangle ||_{W^{0,k_2}_{K,j,3} \times (W^{0,k_2+1}_{K,j,1})^\nu} \leq \epsilon
\]

\[
\leq |||\langle \partial_t^{k_1} u_j(t) | K, \partial_t^{k_1} P_j(t) | |K \rangle - \langle \partial_t^{k_1} u_j(t_0) | K, \partial_t^{k_1} P_j(t_0) | |K \rangle ||_{W^{0,k_2}_{K,j,3} \times (W^{0,k_2+1}_{K,j,1})^\nu} + |||\langle \partial_t^{k_1} u(t_0) | K, \partial_t^{k_1} P(t_0) | |K \rangle - \langle \partial_t^{k_1} u_j(t_0) | K, \partial_t^{k_1} P_j(t_0) | |K \rangle ||_{W^{0,k_2}_{K,j,3} \times (W^{0,k_2+1}_{K,j,1})^\nu} + |||\langle \partial_t^{k_1} u_j(t_0) | K, \partial_t^{k_1} P_j(t_0) | |K \rangle - \langle \partial_t^{k_1} u(t_0) | K, \partial_t^{k_1} P(t_0) | |K \rangle ||_{W^{0,k_2}_{K,j,3} \times (W^{0,k_2+1}_{K,j,1})^\nu} \leq \epsilon
\]

Thus possibly passing to a subsequence for $t \in I$

\[
\langle \partial_t^{k_1} u_j(t) | K, \partial_t^{k_1} P_j(t) | |K \rangle \to \langle \partial_t^{k_1} u(t) | K, \partial_t^{k_1} P(t) | |K \rangle \quad (j \to \infty).
\]

Since $K$ is compact the above formula also shows that the convergence is uniform. Observe that $k_1, k_2$ are arbitrary.

Let $n \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^3$. Let $\Gamma_{\Omega,n}:=\Gamma^{k_2} (\Omega)^n$ and let

\[
(\Gamma_{\Omega,1}^{k_2+1})^\nu := \{ v | v = \nabla w \ (\exists w \in \Gamma_{\Omega,1}^{k_2+1}) \}.
\]

By diagonal argument possibly passing to a subsequence there exists an increasing sequence $\{K_j\} \subset \mathbb{R}^3$ of compact sets such that

\[
|||\langle \partial_t^{k_1} u_j(t) | K, \partial_t^{k_1} P_j(t) | |K \rangle - \langle \partial_t^{k_1} u(t) | K, \partial_t^{k_1} P(t) | |K \rangle ||_{W^{0,k_2}_{K,j,3} \times (W^{0,k_2+1}_{K,j,1})^\nu} \to 0 \quad (j \to \infty).
\]

By assumption passing to a subsequence there exists $c \geq 0$ such that

\[
|||\langle \partial_t^{k_1} u_j(t) | K, \partial_t^{k_1} P_j(t) | |K \rangle ||_{W^{0,k_2}_{K,j,3} \times (W^{0,k_2+1}_{K,j,1})^\nu} \to c \quad (j \to \infty).
\]

On the other hand, since $|||\langle \partial_t^{k_1} u_j(t) | K, \partial_t^{k_1} P_j(t) | |K \rangle ||_{W^{0,k_2}_{K,j,3} \times (W^{0,k_2+1}_{K,j,1})^\nu}$ is bounded, for $(u', P') \in W_3 \times (W_1)$ there exists $M_1 > 0$ such that

\[
| < \langle \partial_t^{k_1} u_j(t) | K, \partial_t^{k_1} P_j(t) | |K \rangle, (u'(t), P'(t)) ||_{W^{0,k_2}_{K,j,3} \times (W^{0,k_2+1}_{K,j,1})^\nu} > M_1 |||(u'(t), P'(t))||_{W^{0,k_2}_{K,j,3} \times (W^{0,k_2+1}_{K,j,1})^\nu} \to 0 \quad (j \to \infty)
\]

so that

\[
< \langle \partial_t^{k_1} u_j(t) | K, \partial_t^{k_1} P_j(t) | |K \rangle, (u'(t), P'(t)) ||_{W^{0,k_2}_{K,j,3} \times (W^{0,k_2+1}_{K,j,1})^\nu} \to 0 \quad (j \to \infty).
\]

44
Thus

\[ ||(\partial_t^{k_1} u_j(t), \partial_t^{k_1} P_j(t)) - (\partial_t^{k_1} u(t), \partial_t^{k_1} P(t))||_{H^{k_2}_{k_j, 3} \times (H^{k_2+1}_{k_j, 1})} \to c^2 \quad (328) \]

\[ \to c^2 + ||(\partial_t^{k_1} u(t), \partial_t^{k_1} P(t))||_{H^{k_2}_{k_j, 3} \times (H^{k_2+1}_{k_j, 1})} \]

\[ - 2 < (\partial_t^{k_1} u(t), \partial_t^{k_1} P(t)), (\partial_t^{k_1} u(t), \partial_t^{k_1} P(t)) >_{H^{k_2}_{k_j, 3} \times (H^{k_2+1}_{k_j, 1})} \]

\( (j \to \infty). \)  

(331)

Since

\[ ||(\partial_t^{k_1} u_j(t), \partial_t^{k_1} P_j(t)) - (\partial_t^{k_1} u(t), \partial_t^{k_1} P(t))||_{H^{k_2}_{k_j, 3} \times (H^{k_2+1}_{k_j, 1})} \to 0 \quad (j \to \infty) \]

it follows that

\[ c = ||(\partial_t^{k_1} u(t), \partial_t^{k_1} P(t))||_{H^{k_2}_{k_j, 3} \times (H^{k_2+1}_{k_j, 1})}. \]  

(333)

Finally

\[ ||(\partial_t^{k_1} u_j(t), \partial_t^{k_1} P_j(t)) - (\partial_t^{k_1} u(t), \partial_t^{k_1} P(t))||_{H^{k_2}_{k_j, 3} \times (H^{k_2+1}_{k_j, 1})} \to c^2 \quad (334) \]

\[ \to c^2 + ||(\partial_t^{k_1} u(t), \partial_t^{k_1} P(t))||_{H^{k_2}_{k_j, 3} \times (H^{k_2+1}_{k_j, 1})} \]

\[ - 2 < (\partial_t^{k_1} u(t), \partial_t^{k_1} P(t)), (\partial_t^{k_1} u(t), \partial_t^{k_1} P(t)) >_{H^{k_2}_{k_j, 3} \times (H^{k_2+1}_{k_j, 1})} \]

\( = 0 \)  

(337)

\( (j \to \infty), \)  

(338)

and \( \{q_j\} \) is convergent to 0 with respect to the ordinary topology. The assertion follows.

The set \( X \) equipped with the topology induced by \( p_k \) is, if exists, denoted by \( X^w \).

**Lemma 85.** Let \( \mathcal{X}_p \) be a sufficiently small convex neighbourhood of \( p \in U_p \) with respect to the ordinary topology identified with the corresponding neighbourhood \( 0 \in T_0 U_p(= T_p \mathcal{X}_p) \) \( (q \in \mathcal{X}_p) \). Then for any sequence \( \{q_j\} \subset \mathcal{X}_p^w \) convergent to \( 0 \in \mathcal{X}_p^w \) is convergent to \( 0 \in \mathcal{X}_p \) with respect to the ordinary topology.

**Proof.** Observe that by Lemma 70 \( \Gamma^\infty(\mathbb{R}^3) = H^\infty(\mathbb{R}^3) \). Assume possibly after passing to a subsequence that \( q_j \not\in \mathcal{X}_p^w (0 < c < 1) \) for large \( j \). \( \mathcal{X}_p \) is sufficiently small so that by Lemma 84 there exists a subsequence \( \{q_{j_l}\} \) convergent with respect to the ordinary topology. Let \( W^1, \ldots, W^m \) be neighbourhoods of \( p \in U_p \) with respect to the topology induced by \( p_k \) and \( \hat{W}^1, \ldots, \hat{W}^m \) their closures with respect to this topology. Let \( \mathcal{X}_p^* \) be the closure of \( \mathcal{X}_p \) with respect to the ordinary topology. Then since \( q_{j_l} \in W^1 \cap \cdots \cap W^m \cap (\mathcal{X}_p \cap c \mathcal{X}_p) \) for large \( l \) and \( W^1, \ldots, W^m \) are arbitrary and since from above \( (\mathcal{X}_p^* \cap c \mathcal{X}_p^* \) is
compact with respect to (the ordinary topology and so) the topology induced by \( p_k \) an elementary argument shows that

\[
\bigcap_W (W \setminus c\mathcal{X}_p) \neq \emptyset.
\]  

(339)

This contradicts with the definition of the topology of \( \mathcal{U}_p \) induced by \( p_k \). Thus \( q_j \in c\mathcal{X}_p \) for large \( j \). Since \( c \) is arbitrary \( \{q_j\} \subset \mathcal{X}_p \) is convergent to \( 0 \in \mathcal{X}_p \) with respect to the ordinary topology. The assertion follows. \( \square \)

The integral with respect to the topologies induced by \( p_k \) (resp. with respect to the ordinary topology of \( \mathcal{U}_p \) and the topology of \( \mathcal{V}_{\Phi(p)} \) induced by \( p_k \)) is denoted by \( \int \) (resp. \( (\int') \)). See Definition 37.

**Lemma 86.** There exist a sufficiently small convex neighbourhood \( \mathcal{X}_p \) of \( p \in \mathcal{U}_p \) with respect to the ordinary topology and a sufficiently small neighbourhood of \( 0 \in (T_{\Phi(p)}\mathcal{V})' \) identified with a sufficiently small neighbourhood \( \mathcal{V}_{\Phi(p)} \) of \( \Phi(p) \in \mathcal{V}_{\Phi(p)} \) such that the multi-valued map \( q \in \mathcal{X}_p \rightarrow \int_0^1 d\Phi(q') \in \mathcal{V}_{\Phi(p)} \) has a continuous branch.

**Proof.** Since \( \mathcal{X}_p \) is sufficiently small the map \( d\Phi(q) \) \( (q \in \mathcal{X}_p) \) from \( T_q \mathcal{X}_p \) to \( \mathcal{V}_{\Phi(p)} \) is continuous and the function \( q \rightarrow (\int_0^1)^*d\Phi(q')||_{(T_{\Phi(p)}\mathcal{V}_{\Phi(p)})} \) is well-defined. Let \( \gamma_q(t) := (1-t)p + tq \ (q \in \mathcal{X}_p) \). Since \( d\Phi \) is well-defined and a continuous map from \( \mathcal{X}_p \) to \( \mathcal{V}_{\Phi(p)} \) there exists \( M > 0 \) such that

\[
\lim_{h \to 0} ||\frac{1}{h} \left( \int_{\gamma_q(t_1)}^{\gamma_q(t_1+h)} d\Phi(q') \right)||_{(T_{\Phi(p)}\mathcal{V}_{\Phi(p)})} = \lim_{h \to 0} ||\frac{1}{h} \left( \int_0^1 d\Phi((1-t)\gamma_q(t_1) + t\gamma_q(t_1 + h))\right)(\gamma_q(t_1 + h) - \gamma_q(t_1))dt||_{(T_{\Phi(p)}\mathcal{V}_{\Phi(p)})} \leq M. \]

(340)

(341)

(342)

(343)

By Theorem 39 there exists an integrable function \( S_{\gamma_q} \) (see Definition 37) such that

\[
\left( \int_{\gamma_q(t_1)}^{\gamma_q(t_2)} d\Phi(q') \right) = \int_{t_1}^{t_2} S_{\gamma_q} dt, \]

(344)

where \( 0 \leq t_1 \leq t_2 \leq 1 \). Then by Lemma 85 an easy argument shows that the map \( q \in \mathcal{X}_p \rightarrow \int_0^1 S_{\gamma_q}(t)dt \in \mathcal{V}_{\Phi(p)} \) gives a desired branch. The assertion follows. \( \square \)

**Lemma 87.** There exist a sufficiently small convex neighbourhood \( \mathcal{X}_p \) of \( p \in \mathcal{U}_p \) with respect to the ordinary topology and a sufficiently small neighbourhood
of $0 \in (T_{\Phi(p)}\mathcal{Y})^w$ identified with a sufficiently small neighbourhood $\mathcal{Y}_s^w(p)$ of $\Phi(p) \in \mathcal{Y}_s^w$ such that

$$\Phi(q) = \Phi(p) + \int_p^q d\Phi(q') \quad (q \in X_p^w, \Phi(q) \in \mathcal{Y}_s^w)$$

(345)

holds for a well-defined branch of $q \in X_p^w \mapsto \int_p^q d\Phi(q') \in \mathcal{Y}_s^w$.

Proof. By Lemma 86 the multi-valued map $q \in X_p^w \mapsto \int_p^q d\Phi(q') \in \mathcal{Y}_s^w$ has a continuous branch so that the function

$$\Phi'(q) := \Phi(p) + \int_p^q d\Phi(q')$$

(346)

has a well-defined branch. Note that since $\Phi'$ has the derivative a.e. along $\gamma_q$ and since the topology of $X_p^w$ induced by $p_k$ is weaker than the ordinary one it is obtained that (*) the derivatives of $\Phi'$ a.e. along $\gamma_q$ with respect to the both topologies at $q$ are equal to $d\Phi(q)$ a.e. in the path. On the other hand it is true that

$$\Phi(q) = \Phi(p) + (\int_p^q)^* d\Phi(q').$$

(347)

Then $\Phi$ and $\Phi'$ satisfy the same conditions: (i) $\Xi(p) = \Phi(p)$ and (ii) the derivative of $\Xi(q)$ a.e. along $\gamma_q$ with respect to the ordinary topology of $X_p$ and the topology of $X_s^w(p)$ induced by $p_k$ coincides with $d\Phi(q)$ a.e. in the path. It follows that $\Phi(q) = \Phi'(q)$. By (*) it is obtained for $t \in [0, 1]$ that

$$S_{\gamma_q}(t) = d\Phi(\gamma_q(t))\gamma_q(t)$$

(348)

(if LHS exists and is equal to $d(\Phi'(\gamma_q(t)))$). Thus for this branch

$$\Phi(q) = \Phi(p) + \int_p^q d\Phi(q').$$

(349)

The assertion follows.

\[\Box\]

$X_p$ and $X_p^2$ are identified with the corresponding neighbourhoods of $0 \in T_qX_p$ ($q \in X_p$) and $0 \in T_QT_qX_p$ ($Q \in X^2_p$).

Lemma 88. Shrink $X_p$ if necessary. Then $d\Phi : (X_p^2)_p^w \to (T\mathcal{Y}_s^w(p))^w$ is continuous and the Fréchet derivative of $\Phi : X_p^w \to \mathcal{Y}_s^w(p)$ (see Definition 40) is equal to $d\Phi$ on $(X_p^2)_p^w$.

Proof. By Lemma 85 it is proved that $d\Phi$ is continuous on $(X_p^2)_p^w$. From this and Lemma 87 the assertion follows easily.

\[\Box\]

Lemma 89. Shrink $X_p$ if necessary. Then $d(d\Phi) : (X_p^4)_p^w \to (TT\mathcal{Y}_s^w(p))^w$ is continuous and the Fréchet derivative of $d\Phi : (X_p^2)_p^w \to (T\mathcal{Y}_s^w(p))^w$ (see Definition 40) is equal to $d(d\Phi)$ on $(X_p^2)_p^w$. 

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Proof. This is proved as in Lemma 88. □

Lemma 90. Let \(L(T_p X_p)^w, (T_{\Phi(p)} Y_{\Phi(p)})^w\) be the set of continuous linear operators from \((T_p X_p)^w\) to \((T_{\Phi(p)} Y_{\Phi(p)})^w\). Then there exists \(M' > 0\) such that
\[
||d\Phi(q') - d\Phi(q)||_{L((T_p X_p)^w, (T_{\Phi(p)} Y_{\Phi(p)})^w)} \leq M'||q' - q||_{X_p^w},
\]
for \(q, q' \in X_p^w\), and such that
\[
||\Phi(q') - \Phi(q) - d\Phi(q)(q' - q)||_{Y_{\Phi(p)}^w} \leq M'||q' - q||_{X_p^w}^2,
\]
for \(q, q' \in X_p^w\).

Proof. By Lemma 89 \(d(d\Phi) : (X_p^w)^w \to (TT_{\Phi(p)} Y_{\Phi(p)})^w\) is continuous. Shrinking \(X_p^\ast\) if necessary there exists \(M' > 0\) such that for \(Q \in (X_p^w)^w\)
\[
||d(d\Phi)(Q)|| \leq M'.
\]
Let \(\gamma : [0, 1] \to X_p^w\) be a smooth path from \(q\) to \(q'\). Then, by Lemma 89
\[
||d\Phi(q') - d\Phi(q)||_{L((T_p X_p)^w, (T_{\Phi(p)} Y_{\Phi(p)})^w)} = \left\| \int_0^1 \frac{\partial}{\partial t} d\Phi(\gamma(t)) dt \right\|_{L((T_p X_p)^w, (T_{\Phi(p)} Y_{\Phi(p)})^w)} \leq M'||q' - q||_{X_p^w},
\]
and
\[
||\Phi(q') - \Phi(q) - d\Phi(q)(q' - q)||_{Y_{\Phi(p)}^w} = \left\| \int_0^1 \frac{\partial^2}{\partial t^2} \Phi(\gamma(t)) dt \right\|_{Y_{\Phi(p)}^w} \leq M'||q' - q||_{X_p^w}^2.
\]
The assertion follows. □

Take the completions \(U^k_p\) and \(\mathcal{V}^k_{\Phi(p)}\) of \(X_p^w\) and \(Y_{\Phi(p)}^w\) induced from \(p_k\). By Lemma 90, \(d\Phi\) extends to a continuous map, \(\Phi\) extends to a \(C^1\)-map from \(U^k_p\) to \(\mathcal{V}^k_{\Phi(p)}\) and the Fréchet derivative with respect to the topologies of \(U^k_p\) and \(\mathcal{V}^k_{\Phi(p)}\) of the extended \(\Phi\) at \(q\) is equal to the value \(d\Phi(q)\) at \(q\) of the extended \(d\Phi\). \(d\Phi(p)\) extends to a topological isomorphism between the completions \(T_p U_p^k\) and \(T_{\Phi(p)} \mathcal{V}^k_{\Phi(p)}\) of \((T_p X_p)^w\) and \((T_{\Phi(p)} Y_{\Phi(p)})^w\) induced from \(p_k\) so that since \(X_p^\ast\) is sufficiently small \(d\Phi(q)\) \((q \in U^k_p)\) is a topological isomorphism. By Theorem 12 there exist sufficiently small neighbourhoods \(U_p\) and \(V_{\Phi(p)}\) of \(p \in U^k_p\) and \(\Phi(p) \in \mathcal{V}^k_{\Phi(p)}\) such that the extended
\[
\Phi : U_p \rightarrow V_{\Phi(p)}
\]
is an isomorphism.
Remark 91. We have taken $O = \mathcal{U}_p^k$, $X$ the linear hull of $\mathcal{U}_p^k$, $Y$ the linear hull of $\mathcal{V}_{\Phi(p)}^k$, $\xi = \Phi$ and $a_0 = p$.

Observe that $\mathcal{V}_{\Phi(p)}$ is an open set of the completion of $\mathcal{X}$. We thus obtain:

Lemma 92. $\mathcal{V}_{\Phi(p)}$ ($p \in \mathcal{X}$) form a Banach manifold.

Replacing $I$ with an interval $J := [t_0, t_0 + T]$ ($0 \leq t_0 < T$ and $T_0 > 0$ is small) we define $\mathcal{X}^J$, $\mathcal{V}^J$, $\mathcal{U}^J$, $\Phi^J$ etc. in the same way as $\mathcal{X}$, $\mathcal{V}$, $\mathcal{U}$, $\Phi$ etc. The above two (1 and 2) of Navier-Stokes conditions also hold if we replace $I$ with $J$ (see Remark 108).

Remark 93. Navier-Stokes condition 3 (see Remark 108) is satisfied.

We need a lemma.

Lemma 94. If $U^J_{p_1} \cap U^J_{p_2} \neq \emptyset$, it is open in $U^J_{p_1}$.

Proof. Let $p \in U^J_{p_1} \cap U^J_{p_2} \cap \mathcal{X}^J$. The tangent space $T_p U^J_{p_1}$ of $U^J_{p_1}$ at $p$ and the tangent space $T_p U^J_{p_2}$ of $U^J_{p_2}$ at $p$ are by definition given as the completions of $T_p \mathcal{X}^J$ and $\Phi$ is a local diffeomorphism on $U^J_{p_1}$ and $U^J_{p_2}$. Thus $T_p U^J_{p_1} = T_p U^J_{p_2} = (d\Phi^J(p))^{-1}(\text{completion of } \Phi^J(p) \mathcal{X}^J)$, where the completion of $\Phi^J(p) \mathcal{X}^J$ is induced from $\mathcal{X}^J \subset \bigcup_p V^J_{\Phi(p)}$. There exists a canonical isomorphism $\varphi^J_p$ from a neighbourhood $W$ of $p \in \mathcal{X}^J$ to a corresponding neighbourhood $W'$ of 0 in $T_p \mathcal{X}^J$ and $\mathcal{U}^k \cup \mathcal{U}^k_2$ is the completion of $\mathcal{X}_{p_1} \cup \mathcal{X}_{p_2}$ induced from $p_k$. Extend $\varphi^J_p|_W$ to $W^1$ (resp. $W^2$), where $W^1$ (resp. $W^2$) is the closure of $W$ in $U^J_{p_1}$ (resp. $U^J_{p_2}$), to obtain a homeomorphism $\varphi^J_{p,1}$ (resp. $\varphi^J_{p,2}$). By definition $\varphi^J_{p,1}(W^1)$ (resp. $\varphi^J_{p,2}(W^2)$) is the closure of $W'$ in $T_p U^J_{p_1}$ (resp. $T_p U^J_{p_2}$). Since $T_p U^J_{p_1} = T_p U^J_{p_2}$ it follows that $\varphi^J_{p,1}(W^1) = \varphi^J_{p,2}(W^2)$ and since $U^J_{p_1} \cup U^J_{p_2} \subset \mathcal{U}^k \cup \mathcal{U}^k_2$, from the above, an easy argument shows that $(\varphi^J_{p,1})^{-1} = (\varphi^J_{p,2})^{-1}$. In particular $(\varphi^J_{p,2})^{-1}(W^2) \cap U^J_{p_1}$ is open in $U^J_{p_1}$. By the definition of the topology $U^J_{p_1} \cap U^J_{p_2} \cap \mathcal{X}^J$ is dense in $U^J_{p_1} \cap U^J_{p_2}$.

Let $p' \in U^J_{p_1} \cap U^J_{p_2}$ be an inner point of $U^J_{p_2}$. Take a small neighbourhood $N$ of $p'$ in $U^J_{p_2}$ and consider $U^J_{p_1} \cap N \cap \mathcal{X}^J$, which is by the same argument as above open in $U^J_{p_1} \cap \mathcal{X}^J$, that is, of the form $O \cap U^J_{p_1} \cap \mathcal{X}^J$, where $O$ is an open set in $U^J_{p_1}$. Take the closure of $U^J_{p_1} \cap N \cap \mathcal{X}^J$. Then the set of all inner points of this closure is open in $U^J_{p_1}$ and the boundary of the closure does not contain $p'$. Hence $U^J_{p_1} \cap U^J_{p_2}$ is open in $U^J_{p_1}$. The assertion follows.

Corollary 95. $U^J_p$ ($p \in \mathcal{X}^J$) form a Banach manifold.

We shall prove the local existence and uniqueness of a sufficiently smooth solution of equation (276) (of which smoothness depends on $k \in \mathbb{N}$).
Let \( t_0 \in I \). Introduce to the inductive limits \( \mathcal{V}_{t_0} \) of \( \bigcup_{J} U_{J}^{p} \) for \( J \ni t_0 \) and \( \mathcal{C}_{t_0} \), of

\[
\mathcal{X}^{(k),J} := \{ (f, r_2, u_0) \in (W_{3}^{k,k})^{J} \times ((W_{3}^{k,k+2})_{\text{div}})^{J} \times \Gamma^{k+2}(\mathbb{R}^{3})^{3} \mid \nabla \cdot u_0 = r_2(t_0) \} \quad (360)
\]

for \( J \ni t_0 \) the natural topologies (the quotient topologies induced from the maps \( \prod_{J} U_{J}^{p} \rightarrow \mathcal{V}_{t_0} \) and \( \prod_{J} \mathcal{X}^{(k),J} \rightarrow \mathcal{C}_{t_0} \)). We first prove the following lemma.

**Lemma 96.** Let \( r := (f, r_2, u_0) \in \mathcal{X}^{J} \). There exists \((u, \nabla p) \in \mathcal{X}^{J}\) such that

\[
\begin{aligned}
\left\{ 
(u + (u \cdot \nabla)u - \nu \Delta u + \nabla p)_{|t = t_0} &= f(t_0), \\
(\nabla \cdot u)_{|t = t_0} &= r_2(t_0), \\
u_{|t = t_0} &= u_0, \\
(\nabla \cdot \dot{u})_{|t = t_0} &= \dot{r}_2(t_0).
\end{aligned}
\]

**Proof.** Since

\[
(\dot{u} + (u \cdot \nabla)u - \nu \Delta u + \nabla p)_{|t = t_0} = f(t_0),
\]

by assumption

\[
-\dot{u}_{|t = t_0} - \nabla p(t_0) = (u \cdot \nabla)u_0 - \nu \Delta u_0 - f(t_0).
\]

By \((\nabla \cdot \dot{u})_{|t = t_0} = \dot{r}_2(t_0)\), \(-\dot{u}_{|t = t_0}\) is determined up to \( \text{Ker}(\nabla \cdot (\cdot)) \). Then by Corollary 71 it is confirmed that there exists such \((u, \nabla p)\). The assertion follows.

\[
\{ \Phi^{J} \}_{J} \text{ induces a map } \Phi_{t_0} : \mathcal{V}_{t_0} \rightarrow \mathcal{C}_{t_0}.
\]

**Lemma 97.** The induced map \( \Phi_{t_0} : \mathcal{V}_{t_0} \rightarrow \mathcal{C}_{t_0} \) is a homeomorphism.

**Proof.** \( \Phi^{J} : \bigcup_{J} U_{J}^{p} \rightarrow \bigcup_{J} V_{J}^{(p)} \) is a local diffeomorphism so that by definition \( \Phi_{t_0} : \mathcal{V}_{t_0} \rightarrow \mathcal{C}_{t_0} \) is a local homeomorphism. \( \mathcal{V}_{t_0} \) is connected (since \( \mathcal{X}^{J} \) is connected and each connected component of \( U_{J}^{p} \) intersects with \( \mathcal{X}^{J} \)) and \( \mathcal{C}_{t_0} \) is simply connected. We claim that \( \Phi_{t_0} \) is surjective. Then \( \Phi_{t_0} \) is a homeomorphism. By Lemma 96 it is proved that for any \( r := (f, r_2, u_0) \in \mathcal{X}^{J} \) there exists a function \( q' \in \bigcup_{J} U_{J}^{p} \) such that \( \Phi^{J}(q')_{|t = t_0} = r(t_0) \), where \( r^{*}|_{t} = r^{*}(t) := (f^{*}(t), r^{*}(t), u_0) \) for \( r^{*} := (f^{*}, r^{*}_2, u_0) \in \mathcal{X}^{J} \). Since \( \Phi^{J} \) is an open map, there exists a deformation \( q \in \bigcup_{J} U_{J}^{p} \) of \( q' \) such that \( \Phi^{J}(q)|_{t} = r(t) \) on a neighbourhood of \( t_0 \). More precisely there exists sufficiently small \( \epsilon > 0 \) such that any \( r^{*} \in \mathcal{X}^{J} \) with

\[
d(r^{*}, \Phi^{J}(q'))_{W_{3}^{k}(\mathbb{R}^{3})^{3}, \Gamma^{k+2}(\mathbb{R}^{3})^{3}} < \epsilon
\]

\[
(365)
\]
(distance) is in $\text{Im}\Phi^J$. Let
\[
(\Gamma^\infty(\mathbb{R}^3)^3)_{\text{div}} := \{ v \mid v = \nabla \cdot w \ (\exists w \in \Gamma^\infty(\mathbb{R}^3)^3) \}.
\]
(366)

It is obtained for small $t_1 > 0$,
\[
d(r(t), \Phi^J(q')|_{t})_{\Gamma^\infty(\mathbb{R}^3)^3 \times (\Gamma^\infty(\mathbb{R}^3)^3)_{\text{div}} \times (\Gamma^\infty(\mathbb{R}^3)^3)_{\text{div}}} < \epsilon \ (t_0 \leq t < t_0 + t_1)
\]
(367)
(distance). By formula (365) and formula (367) there exists $q \in \bigcup_p U^J_p$ such that $\Phi^J(q)|_{t} = r(t)$ $(t_0 \leq t < t_0 + t_1)$. So the image of $\tilde{\Phi}_{t_0}$ contains the inductive limit $\mathscr{C}^\infty_{t_0}$ of $\mathscr{D}^J$ for $J \ni t_0$. Observe that the completion of $\mathscr{D}^J$ in $\mathscr{D}^{J(k),J}$ coincides with the latter space so that the completion of $\mathscr{C}^\infty_{t_0}$ in $\mathscr{C}^J_{t_0}$ does with $\mathscr{C}^J_{t_0}$. From these the assertion is confirmed. Thus $\tilde{\Phi}_{t_0}$ is a homeomorphism. □

Remark 98. Navier-Stokes condition $f$ (see Remark 108) is satisfied.

We obtain a bijection $\tilde{\Phi} : \mathcal{U} := \coprod_{t_0} \mathcal{U}_{t_0} \rightarrow \mathcal{C} := \coprod_{t_0} \mathcal{C}_{t_0}$ and introducing a sheaf structure to $\mathcal{C}$ induced from $\mathcal{U}$ through this bijection a sheaf isomorphism, where the topology of $I$ is generated by $J$’s. We thus conclude as follows.

Corollary 99. $\tilde{\Phi} : \mathcal{U} \rightarrow \mathcal{C}$ is a sheaf isomorphism.

Let $(f, r_2, u_0) \in \mathscr{D}^{J(k),J}$. Then there exists a section $\tilde{p} := (u, \nabla p)$ of $\mathcal{U}$ such that $\tilde{\Phi}(\tilde{p}) = (f, r_2, u_0)$ on $I_1$, where $I_1 := [0, T_1)$ ($0 < T_1 \leq T$) (or $I_1 := [0, T]$) is maximal and $\tilde{p}$ is locally unique in this topology of $I_1$ because $\tilde{\Phi}$ is an isomorphism. Hence $\tilde{p}$ is unique. Consider the following equation on $I_1$:
\[
\begin{bmatrix}
\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u + \nabla p \\
\nabla \cdot u \\
\n u|_{t=0}
\end{bmatrix} := \Phi((u, \nabla p)) = \begin{bmatrix}
f \\
r_2 \\
\nu_0
\end{bmatrix}.
\]
(368)

We say $(u, \nabla p)$ satisfies equation (276) on $I_1$ if it satisfies equation (368). Now we proved the following.

Theorem 100. Let $\nu > 0$. Let $k \in \mathbb{N}$. Let $I := [0, T]$ ($T > 0$). Let
\[
(f, r_2, u_0) \in W^{k,k}_3 \times (W^{k,k+2}_3)_{\text{div}} \times \Gamma^{k+2}(\mathbb{R}^3)^3
\]
(369)
such that $\nabla \cdot u_0 = r_2(0)$. Then there exists an unique $(u, \nabla p)$ satisfying equation (276) on $I_1 := [0, T_1)$ ($0 < T_1 \leq T$) (or on $I_1 := [0, T]$) and that $I_1$ is maximal.

We introduced to $\mathcal{U}$ the relative topology induced from (299) (which depends on $k$) and defined the Banach manifold $\bigcup_p U^J_p$ in Corollary 95 and the sheaf $\mathcal{U}$ after the proof of Lemma 97. Now we remark the following:

Remark 101. In fact $(u, \nabla p) \in \mathcal{U}(I_1)$.
We are going to prove Theorem 78. That $\mathcal{Y} = \mathcal{Z}$ is proved by Navier-Stokes conditions (see Remark 108). Since $(f, 0, u_0) \in A$ and $\mathcal{Y} = \Phi(\mathcal{X})$ there exists $p \in \mathcal{X}$ such that $\Phi(p) = (f, 0, u_0)$, which defines a smooth solution of equation (276). We formalize this in the following way.

Let $\Omega \subset \mathbb{R}^3$ be a subset. Let
\begin{align*}
W_{\Omega,n} := W_{\Omega,n}^{\infty,\infty}, \\
(W_{\Omega,1})_\nabla := \{v \mid v = \nabla w (\exists w \in W_{\Omega,1})\}, \\
(W_{\Omega,3})_{\text{div}} := \{v \mid v = \nabla \cdot w (\exists w \in W_{\Omega,3})\}, \\
\mathcal{X} := W_{\Omega,3} \times (W_{\Omega,1})_\nabla.
\end{align*}

Lemma 102. Let $(f, r_2, u_0) \in \mathcal{Z}$. Let $x \in \mathbb{R}^3$. Then there exists $(u^x, \nabla p^x) \in \mathcal{X}_{\{x\}}$ satisfying equation (276) on $I \times \{x\}$.

Proof. The assertion easily follows. \hfill \square

Lemma 103. Let $(f, r_2, u_0) \in \mathcal{Z}$. Let $x \in \mathbb{R}^3$. Then there exist a compact neighbourhood $K_x$ of $x$ and $(u^{K_x}, \nabla p^{K_x}) \in \mathcal{X}_{K_x}$ satisfying equation (276) on $I \times K_x$.

Proof. By Lemma 102 it follows that there exists $(u^x, \nabla p^x) \in \mathcal{X}_{\{x\}}$ satisfying equation (276) on $I \times \{x\}$. Extend it arbitrarily to $I \times \mathbb{R}^3$ to obtain $(u', \nabla p') \in \mathcal{X}$. Note that
\begin{align*}
d((f, r_2, u_0), \Phi((u', \nabla p')))_{W_{(I_0,3)} \times (W_{(I_0,3)} \times \Gamma_{\mathbb{R}^3})}^{\infty,\infty} (374)
\end{align*}
(distance) is small for any $z$ around $x$. Then since by Lemma 30 $\Phi : \mathcal{X} \to \mathcal{Y}$ is an open map there exist a compact neighbourhood $K_x$ of $x$ and $(u^{K_x}, \nabla p^{K_x}) \in \mathcal{X}_{K_x}$ satisfying equation (276) on $I \times K_x$. The assertion follows. \hfill \square

Lemma 104. Let $(f, r_2, u_0) \in \mathcal{Z}$. Then there exist a closed neighbourhood $K_{\infty}$ of $\infty$ and $(u^{K_{\infty}}, \nabla p^{K_{\infty}}) \in \mathcal{X}_{K_{\infty}}$ satisfying equation (276) on $I \times K_{\infty}$.

Proof. By definition the limits of the derivatives of an element of $\Gamma^\infty(\mathbb{R}^3)$ as $x \to \infty$ are 0 so that there exists $(u', \nabla p') \in \mathcal{X}$ satisfying equation (276) on $I \times \{\infty\}$. Then there exists a neighbourhood $K_{\infty}$ of $\infty$ such that
\begin{align*}
d((f, r_2, u_0), \Phi((u', \nabla p')))_{W_{K_{\infty},3} \times (W_{K_{\infty},3} \times \Gamma_{K_{\infty}}^{\infty})}^{\infty,\infty} (375)
\end{align*}
(distance) is small. By Lemma 30 it is obtained that $\Phi : \mathcal{X} \to \mathcal{Y}$ is an open map so that there exists a desired $(u^{K_{\infty}}, \nabla p^{K_{\infty}})$. The assertion follows. \hfill \square

Remark 105. Navier-Stokes condition 5 (see Remark 65) is satisfied.

Lemma 106. $\mathcal{Y} = \mathcal{Z}$.
**Proof.** Let \((f, r_2, u_0) \in \mathcal{X}\). By Lemma 103 there exists a family \(\{(u^{K_x}, \nabla p^{K_x})\}_x\) such that each \((u^{K_x}, \nabla p^{K_x})\) satisfies equation (276) on \(I \times K_x\). By Lemma 104 there exists \((u^{K_\infty}, \nabla p^{K_\infty})\) in \(\mathcal{I}_{K_\infty}\) satisfying equation (276) on \(I \times K_\infty\). Since \(\mathbb{R}^3 \setminus K_\infty\) is compact we may obtain a finite set \(\{\infty\} \cup \{x_\lambda\}\) such that each \(K_{x_\lambda}\) intersects with another (and \(K_\infty\)) in a set of Lebesgue measure 0 and \(K_\infty \cup \bigcup_{\lambda \in \mathcal{K}_y} K_{x_\lambda} = \mathbb{R}^3\). It follows that there exists \((u, \nabla p) \in (L^2(I \times \mathbb{R}^3)^3)^2\) that is smooth a.e. and satisfies equation (276) a.e. By Lemma 30 and Theorem 100 \(\Phi : \mathcal{X} \to \mathcal{Y}\) is a homeomorphism. Extend \(\Phi^{-1}\) to a continuous map from \(\mathcal{Y}(\subset \mathcal{Z})\) to

\[
\{(u, \nabla p) \in (L^2(I \times \mathbb{R}^3)^3)^2 \mid u, \nabla p \text{ are smooth a.e.}\}.
\]

(376)

Since the set \(\{\infty\} \cup \{x_\lambda\}\) is finite, considering the convolutions \((u_\varepsilon, \nabla p_\varepsilon)\) with mollifiers it is proved that \((u, \nabla p) \in \Phi^{-1}(\mathcal{Y})\) and thus

\[
(u, \nabla p) = \Phi^{-1}((f, r_2, u_0))
\]

(377)

(note that \((u_\varepsilon|_{K_{x_\lambda}}, \nabla p_\varepsilon|_{K_{x_\lambda}})(z = x_\lambda, \infty)\) is convergent in \(\mathcal{I}_{K_{x_\lambda}}\)). Let \((s, y) \in I \times \mathbb{R}^3\) be a singular point of \((u, \nabla p)\). Take another finite decomposition \(K_y \cup K_\infty \cup \bigcup_{\mu \in \mathcal{K}_y} K_{x_{\mu}} = \mathbb{R}^3\) and construct \((u_1, \nabla p_1)\) that is smooth at \((s, y)\) and satisfies equation (276) a.e. Since

\[
(u, \nabla p) = \Phi^{-1}((f, r_2, u_0)) = (u_1, \nabla p_1)
\]

(378)

it follows that \((u, \nabla p)\) is smooth at \((s, y)\). This contradiction shows that \((u, \nabla p) \in \mathcal{X}\). Since

\[
(f, r_2, u_0) = \Phi((u, \nabla p)) \in \mathcal{Y}
\]

(379)

it is concluded that \(\mathcal{X} \subset \mathcal{Y}\). Since the other inclusion is trivial the assertion follows.

\[\square\]

**Remark 107.** *Navier-Stokes conditions (see Remark 108) are used.*

**Proof of Theorem 78.** Since \((f, 0, u_0) \in \mathcal{X} = \mathcal{Y}\) (see Lemma 106) there exists \(p \in \mathcal{X}\) such that \(\Phi(p) = (f, 0, u_0)\), which defines a solution of (276). Further since \(\Phi : \mathcal{X} \to \mathcal{Y}\) is a homeomorphism the solution is unique. The assertion follows. \[\square\]

**Remark 108.** *In the proof of Theorem 78 we used the following Navier-Stokes conditions (which are not assumptions) and conclude \(\Phi\) is a homeomorphism.*

1. \(\Phi : \mathcal{X} \to \mathcal{Y}\) is a \(C^\infty\)-map such that \(d\Phi(p) : T_p \mathcal{X} \to T_{\Phi(p)} \mathcal{Y}\) is a linear isomorphism.
2. Each seminorm $p_k$ of $\mathcal{Y}$, which is induced from the relative topology from

$$W^{k,k}_3 \times (W^{k,k+2}_3)^{\text{div}} \times \Gamma^{k+2}(\mathbb{R}^3)^3,$$

is actually a norm, that is, it satisfies the condition that $p_k(r) = 0$ implies $r = 0$.

3. The above holds if we replace $I$ with $J$.

4. For any $r \in \mathcal{Z}$ there exists $q \in \bigcup_p U^j_p$ such that $\Phi^j(q)|_t = r(t) (t_0 \leq t < t_0 + t_1)$ for small $t_1$.

5. Let $(f, r_z, u_0) \in \mathcal{Z}$. Let $x \in \mathbb{R}^3$. Then there exist a compact neighborhood $K_x$ of $x$ and $(u^{K_x}, \nabla p^{K_x}) \in \mathcal{X}_{K_x}$ satisfying equation (276) on $I \times K_x$. (Further there exist a closed neighborhood $K_\infty$ of $\infty$ and $(u^{K_\infty}, \nabla p^{K_\infty}) \in \mathcal{X}_{K_\infty}$ satisfying equation (276) on $I \times K_\infty$.)

12 Inequality

Theorem 109. Let $\nu > 0$. Let $I := [0, \infty)$. Let $u_0 \in \Gamma^\infty(\mathbb{R}^3)^3$ such that $\nabla \cdot u_0 = 0$. Assume further

$$|\partial_t^\alpha u_0(x)| \leq C_{\alpha,K}(1 + |x|)^{-K}$$

(381)

for each nonnegative integer $K$ and each multi-index $\alpha$. Then there exists an unique $(u, \nabla p) \in W_3 \times (W_1)_\nu$ satisfying the equation

$$\begin{cases}
\frac{\partial u}{\partial t} = -(u \cdot \nabla) u + \nu \Delta u - \nabla p,

\nabla \cdot u = 0,

u|_{t=0} = u_0,

\limsup_{|x| \to \infty} |\partial_t^\alpha u(t, x)| = 0 \ (\forall t, \alpha).
\end{cases}$$

(382)

and $\gamma > 0$, such that for any $t \in I$

$$\int_{\mathbb{R}^3} |u(t, x)|^2 dx \leq \gamma.$$  

(383)

Proof. Observe that $u_0 \in \Gamma^\infty(\mathbb{R}^3)^3$. By Theorem 78 there exists an unique solution of equation (382) on $[0, T]$ for any $T > 0$ and patching the solutions there exists an unique global solution $(u, \nabla p) \in W_3 \times (W_1)_\nu$ of the equation on $I$. Then,
\[
\frac{d}{dt} <u, u>_{L^2(\mathbb{R}^3)}^3
\]
\[
= 2 <\frac{\partial u}{\partial t}, u>_{L^2(\mathbb{R}^3)}^3
\]
\[
= 2 <-(u \cdot \nabla) u + \nu \Delta u - \nabla p, u>_{L^2(\mathbb{R}^3)}^3
\]
\[
= 2 \int_{\mathbb{R}^3} (\text{div} u) \frac{|u|^2}{2} \, dx + 2\nu <\Delta u, u>_{L^2(\mathbb{R}^3)}^3 + 2 \int_{\mathbb{R}^3} p(\text{div} u) \, dx
\]
\[
= 2\nu <\Delta u, u>_{L^2(\mathbb{R}^3)}^3
\]
\[
\leq 0.
\]
Thus for any \( t \in I \)
\[
0 \leq ||u(t, \cdot)||_{L^2(\mathbb{R}^3)}^2 \leq ||u_0||_{L^2(\mathbb{R}^3)}^2 < \infty.
\]
It follows that there exists \( \gamma > 0 \) such that for any \( t \in I \)
\[
\int_{\mathbb{R}^3} |u(t, x)|^2 \, dx \leq \gamma.
\]
The assertion follows. \( \square \)

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References


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