On Pure Motives

Hideto Ishihara

October 10, 2017

e-mail: h.ishihara26@gmail.com

Abstract

Let $X$ be a smooth projective variety over $\mathbb{C}$. Then the pure motive of $X$ of degree $k$ ($k \in \mathbb{Z}$) is the $\mathbb{Q}$-coefficienial de Rham cohomology group $H^k(X, \mathbb{Q})$ of degree $k$.

1 Introduction

In [4] Grothendieck propoeed the concept of motives, ‘a decomposition of an algebraic variety over $\mathbb{C}$ into equidimensional pieces.’ Let $X$ be a smooth projective variety over $\mathbb{C}$. With the use of the theory of Hodge Conjecture the axioms of Weil cohomology theory are formulated. Then we define pure motives, the motives on a smooth projective variety over $\mathbb{C}$, as the $\mathbb{Q}$-coefficienial de Rham cohomology groups $H^k(X, \mathbb{Q})$ ($k \in \mathbb{Z}$).

2 Hodge Conjecture

Let $X$ be a smooth projective variety over $\mathbb{C}$ with a Hodge metric $\omega_0$ (cf. [3], Chapter VII, Section 14, (14.1) Theorem). The de Rham cohomology group $H^k(X, \mathbb{C})$ of degree $k$ decomposes as follows.

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X, \mathbb{C}),$$

(1)

where $H^{p,q}(X, \mathbb{C})$ is the set of de Rham cohomology classes defined by $(p, q)$-forms on $X$ (see [3], Chapter VI, Section 8.2, (8.5) Hodge decomposition theorem). Let $0 \leq p \leq n := \dim X$ be an integer. An analytic $p$-cycle is a finite formal $\mathbb{Q}$-linear combination of irreducible $p$-dimensional complex analytic subvarieties. Let $C_p(X)$ be the set of de Rham cohomology classes defined by analytic $p$-cycles on $X$. Let $H^{2(n-p)}(X, \mathbb{Q})$ be the de Rham cohomology group of degree $2(n-p)$ with coefficients in $\mathbb{Q}$.
Theorem 1.

$$C_p(X) = H^{n-p,n-p}(X, \mathbb{C}) \cap H^{2(n-p)}(X, \mathbb{Q}).$$

(2)

3 Proof of Theorem 1

Let $X$ be a smooth projective variety over $\mathbb{C}$. The following definition, of which expression is slightly changed, is given in [3], Chapter VI, Section 4, (4.1) Definition.

Definition 2. A Kähler metric $\omega_0$ of a smooth complex variety $X$ is a positive definite $(1,1)$-form $\omega_0$ on $X$ such that $d\omega_0 = 0$ on $X$.

The following theorem, of which expression is slightly changed, is given in [3], Chapter VII, Section 14, (14.1) Theorem.

Theorem 3. Let $H^2(X, \mathbb{Q})$ be the de Rham cohomology group of degree 2 with coefficients in $\mathbb{Q}$. A compact complex manifold is a smooth projective variety over $\mathbb{C}$ if and only if there exists a Hodge metric $\omega_0$ on $X$, i.e. there exists a Kähler metric $\omega_0$ such that the de Rham cohomology class $[\omega_0] \in H^2(X, \mathbb{Q})$.

Let $X$ be a smooth projective variety over $\mathbb{C}$ with a Hodge metric $\omega_0$ (cf. Theorem 3). The following, of which expression is slightly changed, is given in [3], Chapter VI, Section 8.2, (8.5) Hodge decomposition theorem.

Theorem 4. Let $H^k(X, \mathbb{C})$ be the de Rham cohomology group of degree $k$. Then

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X, \mathbb{C}),$$

(3)

where $H^{p,q}(X, \mathbb{C})$ is the set of de Rham cohomology classes defined by $(p,q)$-forms on $X$.

Let $0 \leq p \leq n := \dim X$ be an integer. An analytic $p$-cycle is a finite formal $\mathbb{Q}$-linear combination $\sum l \gamma_i$ ($\gamma_i \in \mathbb{Q}$) of irreducible $p$-dimensional complex analytic subvarieties $\{\gamma_i\}$.

Definition 5. Let $Y \subset X$. The restrictions to $Y$ of global analytic $p$-cycles $\Gamma = \sum l \gamma_i$ are $\Gamma \cap Y := \sum l (\gamma_i \cap Y)$.

Definition 6. Let $Y \subset X$. Two restrictions $\Delta_1, \Delta_2$ to $Y$ of global analytic $p$-cycles are equivalent if

$$\int_{\Delta_1} \eta = \int_{\Delta_2} \eta,$$

(4)

for any closed $(p,p)$-form $\eta$ on $X$. Let $C_p(Y)$ be the set of equivalence classes defined by the restrictions to $Y$ of global analytic $p$-cycles.
Remark 7. The set of equivalence classes of analytic $p$-cycles on $X$ is the set of de Rham cohomology classes of analytic $p$-cycles on $X$. Thus the old and the new definitions of $C_p(X)$ coincide.

A complexified analytic $p$-cycle is a formal $\mathbb{C}$-linear combination $\sum c_i \Gamma_i$ ($c_i \in \mathbb{C}$) of irreducible $p$-dimensional complex analytic subvarieties $\{\Gamma_i\}$.

Definition 8. Let $Y \subset X$. The restrictions to $Y$ of global complexified analytic $p$-cycles $\Gamma = \sum c_i \Gamma_i$ are $\Gamma \cap Y := \sum c_i (\Gamma_i \cap Y)$.

Definition 9. Let $Y \subset X$. Two restrictions $\Delta_1, \Delta_2$ to $Y$ of global complexified analytic $p$-cycles are equivalent if

$$\int_{\Delta_1} \eta = \int_{\Delta_2} \eta,$$

for any closed $(p,p)$-form $\eta$ on $X$. Let $C_p(Y) \otimes \mathbb{C}$ be the set of equivalence classes defined by the restrictions to $Y$ of global complexified analytic $p$-cycles.

Let $1 \leq p \leq n-1$ be an integer. Let $B_r(x)$ denote a closed ball with boundary in some coordinate and $\mathcal{U}$ be the set of such balls, that is, $U \in \mathcal{U}$ if there exist $z_1 \in X$ and a coordinate $x_1$ around $z_1$ such that $U = \{ ||x_1|| \leq r_1 \}$ for some $r_1 > 0$. Let

$$\mathcal{U} \big|_{B_r(x)^0} := \{ U \subset B_r(x)^0 \mid x \in U^\circ, U \in \mathcal{U} \}.$$

Definition 10. Two restrictions $\omega_1, \omega_2$ to $B_r(x)^0$ of global harmonic forms are equivalent if

$$\int_{B_r(x)^0} \omega_1 \wedge \eta = \int_{B_r(x)^0} \omega_2 \wedge \eta$$

for any closed form $\eta$ on $X$. Let $H^{*,*}(B_r(x)^0, \mathbb{C})$ be the set of equivalence classes of the restrictions to $B_r(x)^0$ of global harmonic $(*,*)$-forms.

Lemma 11. Let $x \in X$. There exists a closed ball $B_r(x)$ of center $x \in X$ and sufficiently small radius $r > 0$ such that for any point $z_0 \in B_r(x)^0$ and for some coordinate of $B_r(x)^0$ there exist an $(n-p)$-dimensional complex linear subspace $M$ in $B_r(x)^0$ through the origin and a $p$-dimensional complex linear subspace $L$ in $B_r(x)^0$ orthogonal to $M$ through $z_0$ which extends to a global analytic $p$-cycle.

Proof. Embed $X \subset \mathbb{P}^N$ $(N >> 0)$ and consider a closed ball of center $x \in X \subset \mathbb{P}^N$ and sufficiently small radius $r > 0$ in $\mathbb{P}^N$. The intersection of the ball and $X$ is a closed ball $B_r(x)$ in $X$ of center $x \in X$ and radius $r$. Consider hyperplanes $H_1, \ldots, H_{n-p}$ through $z_0$ in $\mathbb{P}^N$. Then since $r > 0$ is sufficiently small there exist such hyperplanes such that $H_1 \cap \cdots \cap H_{n-p} \cap X$ is a global analytic $p$-cycle on $X$ and does not go through $x$ and such that $(H_1 \cap \cdots \cap H_{n-p} \cap X) \cap B_r(x)^0$ is a manifold. Take such global analytic $p$-cycle of the form $H_1 \cap \cdots \cap H_{n-p} \cap X$ as $L$. The remaining statement is easy. The assertion follows. \qed
Lemma 12. \( C_{n-s}(B_r(x)^\circ) \otimes \mathbb{C} \subset H^{s,s}(B_r(x)^\circ, \mathbb{C}). \)

Proof. Extend a complexified \((n-s)\)-cycle \( \Gamma \) to \( X \) and there exists a global harmonic \((s,s)\)-form \( \Omega \) on \( X \) such that

\[
\int_\Gamma \Xi = \int_X \Omega \wedge \Xi
\]

(8)

for any closed \((n-s,n-s)\)-form \( \Xi \) on \( X \). Thus \( [\Gamma]_{|B_r(x)^\circ} = [\Omega]_{|B_r(x)^\circ} \) and the assertion follows. \( \square \)

Definition 13. Let \( V, W \) and \( E \) be vector spaces. A bilinear map \( \Phi : V \times W \to E \) is nondegenerate if

\[
\Phi(v, w) = 0 \quad (\forall w \in W) \Rightarrow v = 0
\]

(9)

and

\[
\Phi(v, w) = 0 \quad (\forall v \in V) \Rightarrow w = 0.
\]

(10)

Let \( [\Gamma] \in C_p(B_r(x)^\circ) \otimes \mathbb{C} \) and \( [\eta] \in H^{p-p}(B_r(x)^\circ, \mathbb{C}) \). Let

\[
\int_\Gamma [\eta] := \{(U, \frac{1}{|U|} \int_U [\Gamma]|_U \wedge [\eta]|_U) \} \quad u \in \mathcal{U}|_{B_r(x)^\circ}.
\]

(11)

Lemma 14. There exists a closed ball \( B_r(x) \) of center \( x \in X \) and sufficiently small radius \( r > 0 \) such that the map

\[
C_p(B_r(x)^\circ) \otimes \mathbb{C} \times H^{p-p}(B_r(x)^\circ, \mathbb{C}) \to \text{Map}(\mathcal{U}|_{B_r(x)^\circ}, \mathbb{C})
\]

(12)

\[
([\Gamma], [\eta]) \mapsto \int_\Gamma [\eta],
\]

(13)

is a nondegenerate bilinear map. Further it suffices to consider \( [\Gamma] \) defined by the restrictions of global analytic \( p \)-cycles of which components are smooth on \( B_r(x)^\circ \).

Proof. By Lemma 11 there exists a closed ball \( B_r(x) \) of center \( x \in X \) and sufficiently small radius \( r > 0 \) such that for any point \( z_0 \in B_r(x)^\circ \) and for some coordinate of \( B_r(x)^\circ \) there exist an \((n-p)\)-dimensional complex linear subspace \( M \) in \( B_r(x)^\circ \) through the origin and a \( p \)-dimensional complex linear subspace \( L \) in \( B_r(x)^\circ \) orthogonal to \( M \) through \( z_0 \) which extends to a global analytic \( p \)-cycle. Let \( [\eta] \in H^{p-p}(B_r(x)^\circ, \mathbb{C}) \). Let \( z_0 \in B_r(x)^\circ \). Divide \( \eta = \alpha_L + \beta \), where \( \eta|_{z_0+r(z_0')} = \alpha_L(z_0') \) for any \( z_0' \) near \( z_0 \). Assume \( \alpha_L(z_0) \neq 0 \) then it is obvious that

\[
\int_{L \cap B(z_0)_r(x)} \eta \neq 0 \quad (\epsilon > 0 \text{ is small}).
\]

(14)

This contradiction shows \( \alpha_L(z_0) = 0 \) and \( \eta(z_0) = \beta(z_0) \). Change coordinates and consider all such \( L \) (cf. the proof of Lemma 11). Combining the resulting
formulas it is, by an elementary argument of exterior algebra, obtained that \( \eta(z_0) = 0 \). Since \( z_0 \in B_r(x)^\circ \) is arbitrary it follows that \([\eta] = 0\). Now it is proved that

\[
\int_{\Gamma}[\eta] = \{(U, \frac{1}{|U|} \int_{\Gamma} [\eta] \wedge |U||u>) \} U \in \mathcal{W}|_{B_r(x)^\circ} = \{(U, 0)\} U \in \mathcal{W}|_{B_r(x)^\circ} \quad (15)
\]

\[
(\forall [\Gamma] \in C_p(B_r(x)^\circ) \otimes \mathbb{C}) \quad (16)
\]

\[
\Rightarrow [\eta] = 0. \quad (17)
\]

By Lemma 12 \( C_{n-p}(B_r(x)^\circ) \otimes \mathbb{C} \subset H^{p,n-p}(B_r(x)^\circ, \mathbb{C}) \) and \( C_p(B_r(x)^\circ) \otimes \mathbb{C} \subset H^{n-p,n-p}(B_r(x)^\circ, \mathbb{C}) \). Thus reversing the roles and considering \([\Gamma]\) corresponding to the restriction to \( B_r(x)^\circ \) of a global harmonic form as an element of \( H^{n-p,n-p}(B_r(x)^\circ, \mathbb{C}) \) it follows that

\[
\int_{\Gamma}[\Gamma'] = \{(U, \frac{1}{|U|} \int_{\Gamma} [\Gamma'] \wedge |U||u>) \} U \in \mathcal{W}|_{B_r(x)^\circ} = \{(U, 0)\} U \in \mathcal{W}|_{B_r(x)^\circ} \quad (18)
\]

\[
(\forall [\Gamma'] \in C_{n-p}(B_r(x)^\circ) \otimes \mathbb{C}) \quad (19)
\]

\[
\Rightarrow [\Gamma] = 0. \quad (20)
\]

The above two show the map (12)-(13) is a nondegenerate bilinear map. Further it suffices to consider \([\Gamma]\) defined by the restrictions of global analytic \( p \)-cycles of which components are smooth on \( B_r(x)^\circ \). The assertion follows. \( \Box \)

The following, of which expression is slightly changed, is given in [3], Chapter VI, Section 3.3, (3.17) Hodge isomorphism theorem.

**Theorem 15.** Let \( X \) be a smooth projective variety over \( \mathbb{C} \) (or, more generally, a compact complex manifold). Let \( \mathcal{H}^k(X, \mathbb{C}) \) be the set of harmonic \( k \)-forms on \( X \). Then \( \mathcal{H}^k(X, \mathbb{C}) \) is finite dimensional and each de Rham cohomology class is uniquely represented by a harmonic form. In particular \( H^k(X, \mathbb{C}) \simeq \mathcal{H}^k(X, \mathbb{C}) \).

Let \( \omega \) be a global harmonic \((n-p,n-p)\)-form on \( X \) such that \([\omega] \in H^{n-p,n-p}(X, \mathbb{C}) \cap H^{2(n-p)}(X, \mathbb{Q}) \). Observe that the set of harmonic forms on \( X \) and thus \( C_p(B_r(x)^\circ) \) and \( H^{n-p}(B_r(x)^\circ, \mathbb{C}) \) are finite dimensional (see Theorem 4 and Theorem 15).

**Definition 16.** The tangent space of a \( C^1 \)-manifold \( \Delta \) at \( x \in \Delta \) is denoted by \( T_x \Delta \). Two finite formal \( \mathbb{C} \)-combinations \( \sum_l c_l \Gamma_l, \sum_{l'} c_{l'} \Gamma'_{l'} \) \((c_l, c_{l'} \in \mathbb{C})\) of \( C^1 \)-manifolds \( \{\Gamma_l\}, \{\Gamma'_{l'}\}\) intersect transversally if \( T_x \Gamma_l \not\subset T_x \Gamma'_{l'} \) and \( T_x \Gamma_l \not\supset T_x \Gamma'_{l'} \) for any \( l, l' \) and for any \( x \in \Gamma_l \cap \Gamma'_{l'} \).

**Lemma 17.** There exist a finite cover \( \{U_\lambda\}_{\lambda=1}^\Lambda \) consisting of closed balls with boundary and \([\Gamma_\lambda] \in C_p(U_\lambda) \otimes \mathbb{C} \) \((1 \leq \lambda \leq \Lambda)\) defined by the restriction of
a global analytic $p$-cycle of which components are smooth on $U_\lambda$ satisfying the following: (i) for any global harmonic $(p, p)$-form $\eta$ on $X$

$$\int_{\Gamma_\lambda} [\eta]|_{U_\lambda} = \int_{U_\lambda} [\omega]|_{U_\lambda} \land [\eta]|_{U_\lambda},$$  \tag{21}$$
and (ii) if $U_\lambda \cap U_\mu \neq \phi$ then for any global harmonic $(p, p)$-form $\eta$ on $X$

$$\int_{\Gamma_\lambda \cap U_\mu} [\eta]|_{U_\lambda \cap U_\mu} = \int_{\Gamma_\mu \cap U_\lambda} [\eta]|_{U_\lambda \cap U_\mu}. \tag{22}$$

Furthermore the above are taken so that $\Gamma_\lambda$ and $\partial U_\lambda$ intersect transversally.

Proof. Since $C_p(B_r(x)^o) \otimes \mathbb{C} \subset H^{n-p, n-p}(B_r(x)^o, \mathbb{C})$ there exists a pairing

$$H^{n-p, n-p}(B_r(x)^o, \mathbb{C}) \times H^{p, p}(B_r(x)^o, \mathbb{C}) \to \text{Map}(\mathcal{U}|_{B_r(x)^o}, \mathbb{C}) \tag{23}$$

extending the map (12)-(13). By Lemma 14 this new pairing is also nondegenerate. Observe that the old pairing is nondegenerate and $H^{n-p, n-p}(B_r(x)^o, \mathbb{C})$ and $H^{p, p}(B_r(x)^o, \mathbb{C})$ are finite dimensional. Thus it follows by linear algebra that $C_p(B_r(x)^o) \otimes \mathbb{C} = H^{n-p, n-p}(B_r(x)^o, \mathbb{C})$.

From above there exists $[\Gamma_x] \in C_p(B_r(x)^o)$ for each $x \in X$ such that for any $U \in \mathcal{U}|_{B_r(x)^o}$

$$\int_{U} [\Gamma_x]|_{U} \land [\eta]|_{U} = \int_{U} [\omega]|_{U} \land [\eta]|_{U} \quad (\forall [\eta] \in H^{p, p}(B_r(x)^o, \mathbb{C})), \tag{24}$$

In particular $[\Gamma_x]$ is such that

$$\int_{B_{r'(x)}} [\Gamma_x]|_{B_{r'(x)}} \land [\eta]|_{B_{r'(x)}} = \int_{B_{r(x)}} [\omega]|_{B_{r(x)}} \land [\eta]|_{B_{r(x)}} \quad (\forall [\eta] \in H^{p, p}(B_r(x)^o, \mathbb{C}) \ (0 < r' < r)). \tag{26}$$

Further $[\Gamma_x]$ is taken to be an equivalence class defined by the restriction of a global analytic $p$-cycle of which components are smooth on $B_r(x)^o$. Thus since $X$ is compact there exist a finite cover $\{U_\lambda\}_{\lambda=1}^\Lambda$ consisting of closed balls with boundary and $[\Gamma_\lambda] \in C_p(U_\lambda) \otimes \mathbb{C} \ (1 \leq \lambda \leq \Lambda)$ defined by the restriction of a global analytic $p$-cycle of which components are smooth on $U_\lambda$ satisfying the following: (i) for any global harmonic $(p, p)$-form $\eta$ on $X$

$$\int_{\Gamma_\lambda} [\eta]|_{U_\lambda} = \int_{U_\lambda} [\omega]|_{U_\lambda} \land [\eta]|_{U_\lambda} \tag{28}$$

and (ii) if $U_\lambda \cap U_\mu \neq \phi$ then for any global harmonic $(p, p)$-form $\eta$ on $X$

$$\int_{\Gamma_\lambda \cap U_\mu} [\eta]|_{U_\lambda \cap U_\mu} = \int_{\Gamma_\mu \cap U_\lambda} [\eta]|_{U_\lambda \cap U_\mu}. \tag{29}$$

Since $\Gamma_\lambda$s are smooth, by shrinking $U_\lambda$s, the above are taken so that $\Gamma_\lambda$ and $\partial U_\lambda$ intersect transversally. The assertion follows. \qed
\{[\Gamma_\lambda]\}_{\lambda=1}^\Lambda defines a de Rham cohomology class \([\omega]\).

**Lemma 18.** There exists an equivalence class defined by the restriction \(\Gamma\) of a global complexified analytic \(p\)-cycle of which components are smooth on \(U_1 \cup \cdots \cup U_\lambda\ such that \([\Gamma] - [\omega]\)|_{U_1 \cup \cdots \cup U_\lambda} = 0 and that \(\Gamma\) intersects with \(\partial(U_1 \cup \cdots \cup U_\lambda)\) transversally.

**Proof.** We prove the assertion by induction on \(\lambda\).

When \(\lambda = 1\), Define \(\Gamma := \Gamma_1\) on \(U_1\) and then \([\Gamma]|_{U_1} = [\Gamma_1] = [\omega]|_{U_1}\). We note that \(\Gamma_1\) is taken so that each component of \(\Gamma_1\) is smooth on \(U_1\) and that \(\Gamma_1\) intersects with \(\partial U_1\) transversally. The assertion follows.

Assume for \(\lambda - 1\) the assertion holds. There exists an equivalence class defined by the restriction \(\Delta\) of a global complexified analytic \(p\)-cycle of which components are smooth on \(U_1 \cup \cdots \cup U_{\lambda-1}\) such that \([\Delta] = [\omega]\)|_{U_1 \cup \cdots \cup U_{\lambda-1}} = 0 and that \(\Delta\) intersects with \(\partial(U_1 \cup \cdots \cup U_{\lambda-1})\) transversally. The restrictions \(\Delta, \Gamma_\lambda\) of global complexified analytic \(p\)-cycles correspond to those \(\Psi, \Phi_\lambda\) of global harmonic \((n - p, n - p)\)-forms. By construction \(([\Psi] - [\omega])|_{U_1 \cup \cdots \cup U_{\lambda-1}} = ([\Delta] - [\omega])|_{U_1 \cup \cdots \cup U_{\lambda-1}} = 0\). It follows that \(\Psi|_{(U_1 \cup \cdots \cup U_{\lambda-1}) \cap \Gamma_\lambda} - \Phi_\lambda|_{(U_1 \cup \cdots \cup U_{\lambda-1}) \cap \Gamma_\lambda} = 0\). Define

\[
\Phi := \begin{cases} 
\Psi \text{ (on } U_1 \cup \cdots \cup U_{\lambda-1}) \\
\Phi_\lambda + (\Psi - \Phi_\lambda) \text{ (on } (U_1 \cup \cdots \cup U_{\lambda-1}) \cap \Gamma_\lambda) \\
\Phi_\lambda \text{ (on } U_\lambda \setminus (U_1 \cup \cdots \cup U_{\lambda-1})).
\end{cases}
\] (30)

By the inductive assumption RHS is well-defined. Let \(\gamma\) be a complex analytic variety appearing in \(\Delta \cap ((U_1 \cup \cdots \cup U_{\lambda-1}) \cap \Gamma_\lambda) - \Gamma_\lambda \cap ((U_1 \cup \cdots \cup U_{\lambda-1}) \cap \Gamma_\lambda)\). Observe that

\[
\int_{\partial((U_1 \cup \cdots \cup U_{\lambda-1}) \cap \Gamma_\lambda)} ([\Psi]|_{(U_1 \cup \cdots \cup U_{\lambda-1}) \cap \Gamma_\lambda} - [\Phi_\lambda]|_{(U_1 \cup \cdots \cup U_{\lambda-1}) \cap \Gamma_\lambda}) \wedge \theta \tag{31}
\]

\[
= \int_{\partial((U_1 \cup \cdots \cup U_{\lambda-1}) \cap \Gamma_\lambda)} 0 \wedge \theta \tag{32}
\]

\[
= 0 \tag{33}
\]

for any \((2p - 1)\)-form \(\theta\) and \(\gamma \cap \partial((U_1 \cup \cdots \cup U_{\lambda-1}) \cap \Gamma_\lambda)\) is of measure 0 on \(\partial((U_1 \cup \cdots \cup U_{\lambda-1}) \cap \Gamma_\lambda)\).

Observe that each component of \(\Delta\) (resp. \(\Gamma_\lambda\)) is smooth on \(U_1 \cup \cdots \cup U_{\lambda-1}\) (resp. \(U_\lambda\)). Recall that, for any \(\mu\), \(\Gamma_\mu\) is taken so that \(\Gamma_\mu\) intersects with \(\partial U_\mu\) transversally (see Lemma 17). Thus \(\Delta\) intersects with \(\partial(U_1 \cup \cdots \cup U_{\lambda-1})\) transversally and \(\Gamma_\lambda\) does with \(\partial U_\lambda\) transversally. If one component \(\delta\) of \(\Delta\) and one component \(\delta_\lambda\) of \(\Gamma_\lambda\) are such that \(T_x \delta \neq T_x \delta_\lambda\) for some \(x \in \delta \cap \delta_\lambda \cap \partial((U_1 \cup \cdots \cup U_{\lambda-1}) \cap U_\lambda)\) then \(\dim(\delta \cap \delta_\lambda) < p\). On the other hand \(\delta \cap \partial(U_1 \cup \cdots \cup U_{\lambda-1})\) and \(\delta_\lambda \cap \partial U_\lambda\) are empty or of real dimension \((2p - 1)\). Thus the above measure property shows a contradiction and the union of the tangent spaces (and the
base spaces) of components of \( \Delta \) on \( \partial((U_1 \cup \cdots \cup U_{\lambda-1}) \cap U_\lambda) \) and that of those of components of \( \Gamma_\lambda \) on \( \partial((U_1 \cup \cdots \cup U_{\lambda-1}) \cap U_\lambda) \) coincide. Further, from this, the above measure property shows that the coefficients of \( \Delta \cap ((U_1 \cup \cdots \cup U_{\lambda-1}) \cap U_\lambda) \) and \( \Gamma_\lambda \cap ((U_1 \cup \cdots \cup U_{\lambda-1}) \cap U_\lambda) \) coincide. Let

\[
\Gamma := \begin{cases} 
\Delta & \text{(on } U_1 \cup \cdots \cup U_{\lambda-1}) \\
\Gamma_\lambda & \text{(on } U_\lambda \setminus (U_1 \cup \cdots \cup U_{\lambda-1})) \end{cases}.
\]

(34)

\( \Gamma \) intersects with \( \partial(U_1 \cup \cdots \cup U_{\lambda-1}) \cap U_\lambda \) transversally so that any component of \( \Gamma \) is locally a graph of \( C^1 \)-functions that are holomorphic outside a proper smooth manifold. By taking limit it is obvious that the functions satisfy Cauchy-Riemann equations and hence any component of \( \Gamma \) is smooth complex analytic on \( U_1 \cup \cdots \cup U_\lambda \). We note that \( \Gamma \) intersects with \( \partial(U_1 \cup \cdots \cup U_\lambda) \) transversally. The assertion follows.

\[\square\]

**Lemma 19.** There exists a natural nondegenerate pairing

\[ H^{2(n-p)}(X, \mathbb{Q}) \times H^{2p}(X, \mathbb{Q}) \to \mathbb{Q}. \]

(35)

**Proof.** \( X \) is a smooth projective variety with the Hodge metric \( \omega_0 \) so that

\[ *1 \in H^{2n}(X, \mathbb{Q}), \]

(36)

where \( * \) is the Hodge’s star operator with respect to \( \omega_0 \). Thus there exists a natural nondegenerate pairing

\[ H^{2(n-p)}(X, \mathbb{Q}) \times H^{2p}(X, \mathbb{Q}) \to \mathbb{Q} \]

(37)

\[ ([\Theta], [\Xi]) \mapsto <[\Theta], [\Xi]>_{\omega_0}, \]

(38)

where

\[ <[\Theta], [\Xi]>_{\omega_0} *1 = [\Theta] \cup [\Xi]. \]

(39)

The assertion follows.

\[\square\]

**Proof of Theorem 1.** It is clear that

\[ C_p(X) \subset H^{n-p,n-p}(X, \mathbb{C}) \cap H^{2(n-p)}(X, \mathbb{Q}). \]

(40)

Thus it suffices to show the other inclusion. Let \([\omega] \in H^{n-p,n-p}(X, \mathbb{C}) \cap H^{2(n-p)}(X, \mathbb{Q})\). \( X = U_1 \cup \cdots \cup U_\lambda \) and by Lemma 18 it is concluded that there exists a de Rham cohomology class \([\Gamma]\) (cf. Theorem 15) defined by a complexified analytic \( \rho \)-cycle such that \([\Gamma] - [\omega] = 0 \) on \( X \). Thus

\[ [\Gamma] = [\omega] \in H^{n-p,n-p}(X, \mathbb{C}) \cap H^{2(n-p)}(X, \mathbb{Q}). \]

(41)

Assume each \([\Gamma_i]\) corresponds to a \( p \)-dimensional irreducible complex analytic subvariety and that \( \{[\Gamma_i]\}_i \) forms a basis of \( C_p(X) \). Express \([\Gamma]\) as

\[ [\Gamma] = \sum_i c_i [\Gamma_i] \quad (c_i \in \mathbb{C}). \]

(42)
Extend \([\{\Gamma_i\}\}_i\) to a basis of \(H^{n-p,n-p}(X, \mathbb{C})\cap H^{2(n-p)}(X, \mathbb{Q})\) and then by Lemma 19, formula (40) and formula (41) it is obtained by linear algebra that \(c_i \in \mathbb{Q}\) (\(\forall i\)). Thus \(\Gamma\) is in fact an analytic \(p\)-cycle on \(X\). The assertion follows. \(\Box\)

**Remark 20.** From [3], Chapter II, Section 8.2, (8.10) Chow’s theorem, the set of analytic \(p\)-cycles on \(X\) and that of algebraic ones coincide.

## 4 Pure motives

Let \(X\) be a smooth projective variety over \(\mathbb{C}\) of dimension \(n\).

**Definition 21.** Let \(\Gamma_1, \Gamma_2\) be two algebraic \(p\)-cycles on \(X\). They are numerically equivalent if

\[
\int_X [\Gamma_1] \wedge [\Delta] = \int_X [\Gamma_2] \wedge [\Delta] \tag{43}
\]

for any \([\Delta] \in C_{n-p}(X)\).

**Theorem 22.** Numerical equivalence and de Rham cohomological equivalence are equivalent.

**Proof.** Let \(\Gamma_1, \Gamma_2\) be two algebraic \(p\)-cycles on \(X\). If \(\Gamma_1\) and \(\Gamma_2\) are de Rham cohomologically equivalent then by definition \(\Gamma_1\) and \(\Gamma_2\) are numerically equivalent. If \(\Gamma_1\) and \(\Gamma_2\) are numerically equivalent then by Theorem 1 they are equivalent under the de Rham cohomological equivalence. The assertion follows. \(\Box\)

**Definition 23.** Let \(\mathcal{P}(\mathbb{C})\) be the category of smooth projective varieties over \(\mathbb{C}\) and \(\mathcal{D}_Q\) the category of bigraded \(\mathbb{Q}\)-vector spaces. A Weil cohomology theory is a contravariant functor \(H^* : \mathcal{P}(\mathbb{C}) \to \mathcal{D}_Q\) such that the following holds.

1. Let \(X \in \mathcal{P}(\mathbb{C})\) be of dimension \(n\) then \(H^{p,q}(X) = 0\) unless \(0 \leq p, q \leq n\).
2. Let \(X \in \mathcal{P}(\mathbb{C})\) be of dimension \(n\) then the Poincaré duality holds, i.e. the following is a nondegenerate pairing:

\[
H^{p,q}(X) \times H^{n-p,n-q}(X) \to \mathbb{Q} \tag{44}
\]

where \(X \in \mathcal{P}(\mathbb{C})\) and \(0 \leq p, q \leq n\).
3. The Küneth formula holds: for \(X, Y \in \mathcal{P}(\mathbb{C})\)

\[
H^{*,*}(X \times Y) \simeq H^{*,*}(X) \otimes H^{*,*}(Y). \tag{45}
\]

4. Let \(X \in \mathcal{P}(\mathbb{C})\) and \(Z_p(X)\) the set of algebraic \(p\)-cycles on \(X\). There exists a cycle map \(\gamma_X : Z_p(X) \to H^{2p}(X)\).
5. Let \(X \in \mathcal{P}(\mathbb{C})\) be of dimension \(n\). Then there exists an isomorphism \(H^{2n}(X) \cong H^{2n}(X, \mathbb{Q})\) compatible with the cycle maps, where \(H^{2n}(X, \mathbb{Q})\) is the de Rham cohomology group of degree \(2n\).
6. Let \(X \in \mathcal{P}(\mathbb{C})\). Then \(\dim_{\mathbb{Q}} H^{p,p}(X) \leq \dim_{\mathbb{Q}} H_{p,p}(X, \mathbb{Q})\), where \(H^{p,p}(X, \mathbb{Q}) := H^{p,p}(X, \mathbb{C}) \cap H^{2p}(X, \mathbb{Q})\).
**Definition 24.** Let $\Gamma_1, \Gamma_2$ be two algebraic $p$-cycles on $X$. They are homologically equivalent if $\gamma_X(\Gamma_1) = \gamma_X(\Gamma_2)$, where $\gamma_X$ is a cycle map to a Weil cohomology group.

**Theorem 25.** Homological equivalence and de Rham cohomological equivalence are equivalent.

**Proof.** Let $\Gamma$ be an algebraic $p$-cycle. If $[\Gamma] \neq 0$, then $0 < [\Gamma] \cup \ast[\Gamma]$ and by Definition 23, (5), the homomorphism $\gamma_X(\Gamma) \cup (\ast \gamma_X(\Gamma)) \mapsto [\Gamma] \cup \ast[\Gamma]$, where $\ast \gamma_X(\Gamma)$ is the Poincaré dual of $\gamma_X(\Gamma)$, is well-defined. Especially $[\Gamma] \neq 0$ implies $\gamma_X(\Gamma) \neq 0$. Thus $\gamma_X(\Gamma) = 0$ implies $[\Gamma] = 0$. Conversely reversing the roles it follows that $[\Gamma] = 0$ implies $\gamma_X(\Gamma) = 0$. Thus homological equivalence and de Rham cohomological equivalence are equivalent. The assertion follows. \qed

**Theorem 26.** Let $X \in P(\mathbb{C})$ be of dimension $n$. Weil cohomology groups $H^{*,*}(X)$ are isomorphic to $H^{*,*}(X, \mathbb{Q})$.

**Proof.** Let $Y \in P(\mathbb{C})$. By Definition 23 (4), Theorem 25 and Theorem 1 it is obtained that $H^{p,p}(X, \mathbb{Q}) \subset H^{p,p}(Y)$ and by Definition 23, (6) it is obtained that $\dim_{\mathbb{Q}} H^{p,p}(Y) \leq \dim_{\mathbb{Q}} H^{p,p}(Y, \mathbb{Q}) < \infty$. Thus $H^{p,p}(Y) \simeq H^{p,p}(Y, \mathbb{Q})$.

Let $G := \{m_1 + im_2 \mid m_1, m_2 \in \mathbb{Z}\}$ and $T := \mathbb{C}/G$. By Definition 23, (5) and (2) it is obtained that $H^{p,p}(T) \simeq H^{p,p}(T, \mathbb{Q})$ ($p = 0, 1$). Consider $H^{1,1}(T^n)$. By Definition 23, (3) and (2) and by an elementary argument it is obtained that $H^{1,0}(T) \simeq H^{1,0}(T, \mathbb{Q})$ and $H^{0,1}(T) \simeq H^{0,1}(T, \mathbb{Q})$. By Definition 23, (3) it follows that $H^{p,q}(T^n) \simeq H^{p,q}(T^n, \mathbb{Q})$.

Observe that
\[
\dim_{\mathbb{Q}} H^{p,q}(T^n, \mathbb{Q}) > 0 \ (0 \leq p, q \leq n),
\]
and that
\[
H^{p,p}(X \times T^n) = \bigoplus_{p_1 + p_2 = p} H^{p_1,q_1}(X) \otimes H^{p_2,q_2}(T) \ (0 \leq p \leq n).
\]
Thus it follows by induction that $H^{p,q}(X) \simeq H^{p,q}(X, \mathbb{Q})$ ($0 \leq p, q \leq n$). The assertion follows. \qed

**Definition 27.** Let $X$ be a smooth projective variety over $\mathbb{C}$. The pure motive of $X$ of degree $k$ ($k \in \mathbb{Z}$) is the de Rham cohomology group $H^k(X, \mathbb{Q})$ of degree $k$.

**References**

claymath.org/sites/default/files/hodge.pdf [Accessed: 18th January 2017]

[Accessed: 14th February 2016]
