Tate Conjecture

Hideto Ishihara

January 8, 2018

e-mail: h.ishihara26@gmail.com

Abstract

Tate Conjecture is a well-known problem. We prove it. A new definition of the cycle map in étale cohomology theory on an arithmetic variety is introduced.

2010 Mathematics Subject Classification 14G40, 14C30

Keywords: Tate Conjecture, Hodge Conjecture, Analytic cycles, Harmonic forms

1 Introduction

Let $X$ be a smooth projective variety over $\mathbb{C}$ with a Hodge metric $\omega_0$ (cf. [2], Chapter VII, Section 14, (14.1) Theorem). The de Rham cohomology group $H^k(X, \mathbb{C})$ of degree $k$ decomposes as follows.

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X, \mathbb{C}),$$  \hfill (1)

where $H^{p,q}(X, \mathbb{C})$ is the set of de Rham cohomology classes defined by $(p, q)$-forms on $X$ (see [2], Chapter VI, Section 8.2, (8.5) Hodge decomposition theorem). Let $0 \leq p \leq n := \dim X$ be an integer. An analytic p-cycle is a finite formal $\mathbb{Q}$-linear combination of irreducible $p$-dimensional complex analytic sub-varieties. Let $C_p(X)$ be the set of de Rham cohomology classes defined by analytic p-cycles on $X$. Let $H^{2(n-p)}(X, \mathbb{Q})$ be the de Rham cohomology group of degree $2(n-p)$ with coefficients in $\mathbb{Q}$. We first prove the following theorem.

Theorem 1.

$$C_p(X) = H^{n-p,n-p}(X, \mathbb{C}) \cap H^{2(n-p)}(X, \mathbb{Q}).$$  \hfill (2)

The result is trivial if $p = 0, n$. By Lefschetz $(1, 1)$ theorem the result is known for $p = 1, n - 1$. (See [4].)

Let $a, b$ be distinct prime numbers. Let $K \supset F_a$ be a field of finite extension

1
of $F_{\alpha}$. Let $V$ be a representation of $\text{Gal}(\overline{\mathbb{Q}}_b/\mathbb{Q}_b)$ over $\mathbb{Q}_b$. Let $X_0$ be a smooth projective scheme over $K$. Let $X_0(K)$ be the set of $K$-rational points of $X_0$. Let $Z_p(X_0(K))$ be the set of formal $V$-linear combinations of irreducible $p$-dimensional reduced subschemes of $X_0(K)$. Let $\mathcal{K}$ be the algebraic separable closure of $K$ and $X_0(\mathcal{K})$ the set of $\mathcal{K}$-rational points of $X_0$. Let $\Omega_{\mathcal{Q}}$ be the ring of integers of $\overline{\mathcal{Q}}$. Let $a \subset \Omega_{\mathcal{Q}}$ be a maximal ideal such that $a \cap \mathcal{Z} = a\mathcal{Z}$. There exists a natural surjective map from the quotient field of $\Omega_{\mathcal{Q}}/a$ to $\mathcal{K}$. For $\sigma \in \text{Gal}(\overline{\mathcal{Q}}/\mathcal{Q})$ such that $\sigma(a) \subset a$ let $\sigma \in \text{Gal}(K/F_{\alpha})$ be the induced element. Let

$$I_a := \{ \sigma \in \text{Gal}(\overline{\mathcal{Q}}/\mathcal{Q}) \mid \sigma(a) \subset a \text{ and } \sigma = \text{id} \}.$$  \hfill (3)

Let $C_{I_a} := \mathcal{Q}^{I_a}$. Let $\iota : K \hookrightarrow \mathcal{Q}^{I_a} \hookrightarrow \mathcal{Q}$. Let $X_0(\mathcal{C})$ be the set of $\mathcal{C}$-rational points of the reduced scheme induced from $X_0(K)$ via $\iota$. Let $(X_0)_{\mathcal{C}} \rightarrow X(\mathcal{C})$ be the resolution of singularity. Let $Z_p((X_0)_{\mathcal{C}})$ be the set of formal $V$-linear combinations of irreducible $p$-dimensional reduced subschemes of $(X_0)_{\mathcal{C}}$. By Theorem 1 it is easy to prove

$$Z_p((X_0)_{\mathcal{C}}) \to H^{n-p,n-p}_{\text{et}}((X_0)_{\mathcal{C}}, V)$$  \hfill (4)

is surjective. Let $C_p((X_0)_{\mathcal{C}})$ be the image of $Z_p((X_0)_{\mathcal{C}})$ under this map. Let $H^{n-p,n-p}_{\text{et}}((X_0)_{\mathcal{C}}, V)$ be the image of the map

$$H^{n-p,n-p}_{\text{et}}((X_0)_{\mathcal{C}}, V) \to H_{\text{et}}((X_0)_{\mathcal{C}}, V) \to H_{\text{et}}(X_0(\mathcal{K}), V).$$  \hfill (5)

Let $\text{Gal}(\mathcal{K}/K)$ act on the set $C_p((X_0)_{\mathcal{C}})^{I_a}$ naturally. This induces an action of $\text{Gal}(\mathcal{K}/K)$ on $H^{n-p,n-p}_{\text{et}}(X_0(\mathcal{K}), V)$ via the surjective map

$$C_p((X_0)_{\mathcal{C}}) \to H^{n-p,n-p}_{\text{et}}((X_0)_{\mathcal{C}}, V) \to H^{n-p,n-p}_{\text{et}}((X_0)_{\mathcal{C}}, V)^{I_a} \to \downarrow$$
$$C_p((X_0)_{\mathcal{C}})^{I_a} \to \downarrow H^{n-p,n-p}_{\text{et}}(X_0(\mathcal{K}), V).$$  \hfill (6)

Let

$$\mathcal{P}_K : \begin{array}{c} Z_p((X_0)_{\mathcal{C}}) \to H^{n-p,n-p}_{\text{et}}((X_0)_{\mathcal{C}}, V) \\
\uparrow \quad \downarrow \\
Z_p(X_0(K)) \quad H^{n-p,n-p}_{\text{et}}(X_0(\mathcal{K}), V)^{\text{Gal}(\mathcal{K}/K)} \end{array}.$$  \hfill (7)

Two elements $\Gamma_1, \Gamma_2$ of $Z_p(X_0(K))$ is numerically equivalent if

$$\mathcal{P}_K(\Gamma_1) \cup \mathcal{P}_K(\Delta) = \mathcal{P}_K(\Gamma_2) \cup \mathcal{P}_K(\Delta)$$  \hfill (8)

for any $\Delta \in Z_{n-p}(X_0(K))$. In this case we write $\Gamma_1 \sim \Gamma_2$. We prove the following well-known theorem (see e.g. [3]).

**Theorem 2 (Tate Conjecture).** The cycle map

$$C_p((X_0)_{\mathcal{C}}) \to H^{n-p,n-p}_{\text{et}}((X_0)_{\mathcal{C}}, V) \to \downarrow Z_p(X_0(K))/\sim \to H^{n-p,n-p}_{\text{et}}(X_0(\mathcal{K}), V)^{\text{Gal}(\mathcal{K}/K)}$$  \hfill (9)

is surjective.
It is proposed by John Tate (published in 1965) and is a central problem of arithmetic algebraic geometry (see [5]).

The proof proceeds as follows:

As above the result is known for \( p = 0, n \). Let \( 1 \leq p \leq n - 1 \) be an integer. We show that

\[
C_p(X) \supset H^{n-p,n-p}(X, \mathbb{C}) \cap H^{2(n-p)}(X, \mathbb{Q}).
\]  

(10)

The other inclusion is trivial. It suffices to show that for any element \( [\omega] \) of RHS there exists a global analytic \( p \)-cycle \( \Gamma \) on \( X \) such that

\[
[\Gamma] = [\omega]
\]

as de Rham cohomology classes on \( X \).

Let \( Y \subset X \). The restrictions to \( Y \) of global analytic \( p \)-cycles \( \Gamma = \sum c_i \Gamma_i \) are \( \Gamma \cap Y := \sum c_i (\Gamma_i \cap Y) \). Two restrictions \( \Delta_1, \Delta_2 \) to \( Y \) of global analytic \( p \)-cycles are equivalent if

\[
\int_{\Delta_1} \eta = \int_{\Delta_2} \eta.
\]

(12)

for any closed \( (p,p) \)-form \( \eta \) on \( X \). Let \( C_p(Y) \) be the set of equivalence classes defined by the restrictions to \( Y \) of global analytic \( p \)-cycles.

A complexified analytic \( p \)-cycle is a finite formal \( \mathbb{C} \)-linear combination of irreducible \( p \)-dimensional complex analytic subvarieties. An equivalence class defined by the restriction to \( Y \) (\( Y \subset X \)) of a global complexified analytic \( p \)-cycle and the set \( C_p(Y) \otimes \mathbb{C} \) of equivalence classes defined by the restrictions to \( Y \) of global complexified analytic \( p \)-cycles are defined similarly.

Let \( 1 \leq p \leq n - 1 \) be an integer. Let \( B_r(x) \) denote a closed ball with boundary in some coordinate and \( \mathcal{U} \) be the set of such balls (the choice of a coordinate also varies). Let

\[
\mathcal{U}|_{B_r(x)} := \{ U \subset B_r(x) \mid x \in U^0, U \in \mathcal{U} \}.
\]

(13)

Two restrictions \( \omega_1, \omega_2 \) of global harmonic forms are equivalent if

\[
\int_{B_r(x)} \omega_1 \wedge \eta = \int_{B_r(x)} \omega_2 \wedge \eta
\]

(14)

for any closed form \( \eta \) on \( X \). Let \( H^{*,*}(B_r(x), \mathbb{C}) \) be the set of equivalence classes of the restrictions to \( B_r(x) \) of global harmonic \((*,*)\)-forms. Let \([\Gamma] \in \[

\]
\[ C_p(B_r(x)^\circ) \otimes \mathbb{C} \text{ and } [\eta] \in H^{p,p}(B_r(x)^\circ, \mathbb{C}). \text{ Let} \]
\[ \int_{\Gamma} [\eta] := \{ (U, \frac{1}{|U|} \int_U [\Gamma]|_U \wedge [\eta]|_U) \} \text{ if } \forall \eta \in \mathbb{U}|_{B_r(x)^\circ}. \]

Then for sufficiently small \( r > 0 \)
\[ C_p(B_r(x)^\circ) \otimes \mathbb{C} \times H^{p,p}(B_r(x)^\circ, \mathbb{C}) \to \text{Map}(\mathbb{U}|_{B_r(x)^\circ}, \mathbb{C}) \]
\[ ([\Gamma], [\eta]) \mapsto \int_{\Gamma} [\eta] \]

is a nondegenerate bilinear map. Indeed it suffices to consider \([\Gamma] \in C_p(B_r(x)^\circ) \otimes \mathbb{C}\) defined by the restrictions of global analytic \( p \)-cycles of which components are smooth on \( B_r(x)^\circ \).

The tangent space of a \( C^1 \)-manifold \( \Delta \) at \( x \in \Delta \) is denoted by \( T_x \Delta \). Two finite formal \( \mathbb{C} \)-combinations \( \sum_l \xi_l \Gamma_l, \sum_{l'} \xi_{l'} \Gamma_{l'} \) (\( \xi_l, \xi_{l'} \in \mathbb{C} \)) of \( C^2 \)-manifolds \( \{ \Gamma_l \} \), \( \{ \Gamma_{l'} \} \) intersect transversally if \( T_x \Gamma_l \not\subset T_x \Gamma_{l'} \) and \( T_x \Gamma_l \not\supset T_x \Gamma_{l'} \) for any \( l, l' \) and for any \( x \in \Gamma_l \cap \Gamma_{l'} \).

Let \( \omega \) be a harmonic \((n-p,n-p)\)-form on \( X \) such that \( [\omega] \in H^{n-p,n-p}(X, \mathbb{C}) \cap H^{2(n-p)}(X, \mathbb{Q}) \). Note that \( X \) is compact and that \( H^{\ast,\ast}(B_r(x)^\circ, \mathbb{C}) \) is finite dimensional. By the above argument there exist a finite cover \( \{ U_\lambda \}_{\lambda=1}^A \) consisting of closed balls with boundary and \([\Gamma_\lambda] \in C_p(U_\lambda) \otimes \mathbb{C} \) \((1 \leq \lambda \leq A)\) defined by the restriction of a global analytic \( p \)-cycle of which components are smooth on \( U_\lambda \) satisfying the following: (i) for any global harmonic \((p,p)\)-form \( \eta \) on \( X \)
\[ \int_{\Gamma_\lambda} [\eta]|_{U_\lambda} = \int_{U_\lambda} [\omega]|_{U_\lambda} \wedge [\eta]|_{U_\lambda}, \]
and (ii) if \( U_\lambda \cap U_\mu \neq \emptyset \) then for any global harmonic \((p,p)\)-form \( \eta \) on \( X \)
\[ \int_{\Gamma_\lambda \cap U_\mu} [\eta]|_{U_\lambda \cap U_\mu} = \int_{\Gamma_\lambda \cap U_\mu} [\eta]|_{U_\lambda \cap U_\mu}. \]

Further since \( \Gamma_\lambda \)'s are smooth, shrinking \( U_\lambda \)'s, the above are taken so that \( \Gamma_\lambda \)
and \( \partial U_\lambda \) intersect transversally.

We construct a global complexified analytic \( p \)-cycle.

\([\{ \Gamma_\lambda \}_{\lambda=1}^A \] defines a de Rham cohomology class \([\omega] \). The restrictions \( \Gamma_\lambda \) of global complexified analytic \( p \)-cycles correspond to those \( \Phi_\lambda \) of global harmonic \((n-p,n-p)\)-forms. Note that \( \Gamma_\lambda \) is defined on a closed ball \( U_\lambda \) and that \( \Gamma_\lambda \) intersects with \( \partial U_\lambda \) transversally. Thus \( \Phi := \Phi_1 \) is inductively extended to \( X \) so that \([\Phi - [\omega]]|_{U_1 \cup \cdots \cup U_\lambda} = 0 \) and there exists a restriction \( \Gamma \) of global complexified analytic \( p \)-cycle such that \( [\Gamma]|_{U_1 \cup \cdots \cup U_\lambda} = [\Phi]|_{U_1 \cup \cdots \cup U_\lambda} \). From \( X = U_1 \cup \cdots \cup U_\lambda \), it is concluded that there exists a de Rham cohomology class \([\Gamma] \) defined by a
global complexified analytic $p$-cycle such that $[\Gamma] - [\omega] = [\Phi] - [\omega] = 0$ on $X$.

We construct a desired analytic $p$-cycle on $X$.

From above it is obtained that for any $[\omega] \in H^{n-p,n-p}(X,\mathbb{C}) \cap H^{2(n-p)}(X,\mathbb{Q})$ there exists $[\Gamma] \in C_p(X) \cap \mathbb{C}$ such that $[\Gamma] = [\omega]$. $X$ is a smooth projective variety over $\mathbb{C}$ with the Hodge metric $\omega_0$ and thus $\Gamma$ is actually an analytic $p$-cycle on $X$. The assertion of Theorem 1 follows.

**Remark 3.** From [2], Chapter II, Section 8.2, (8.10) Chow's theorem, the set of analytic $p$-cycles on $X$ and that of algebraic ones coincide.

Let

$$\mathcal{P}_R : Z_p((X_0)_\mathbb{C}) \to H^{n-p,n-p}_{\text{et}}((X_0)_\mathbb{C},\mathbb{V})$$

$$\uparrow$$

$$Z_p(X_0(\mathbb{K})) \to H^{n-p,n-p}_{\text{et}}(X_0(\mathbb{K}),\mathbb{V}).$$

Two elements $\Gamma_1, \Gamma_2$ of $Z_p(X_0(\mathbb{K}))$ is numerically equivalent if

$$\mathcal{P}_R(\Gamma_1) \cap \mathcal{P}_R(\Gamma_2) = \mathcal{P}_R(\Gamma_1) \cup \mathcal{P}_R(\Gamma_2)$$

for any $\Delta \in Z_{n-p}(X_0(\overline{\mathbb{K}}))$. In this case we write $\Gamma_1 \sim \Gamma_2$. Let $\mathcal{V}$ be the induced reduced subscheme of $X_0(\mathbb{C})$ from a reduced subscheme of $X_0(\mathbb{K})$ via $\iota$. Then no $K$-rational point of $\mathcal{V}$ is in the singular locus of $X_0(\mathbb{C})$. From the definitions it is easy to prove

$$(Z_p(X_0(K))/ \sim) = (Z_p(X_0(\overline{\mathbb{K}}))/ \sim)^{\text{Gal}(\overline{\mathbb{K}}/K)} = (C_p((X_0)_\mathbb{C})^{\text{et}})^{\text{Gal}(\overline{\mathbb{K}}/K)}$$

and

$$H^{n-p,n-p}_{\text{et}}(X_0(\overline{\mathbb{K}}),\mathbb{V})^{\text{Gal}(\overline{\mathbb{K}}/K)} = (H^{n-p,n-p}_{\text{et}}((X_0)_\mathbb{C},\mathbb{V})^{\text{et}})^{\text{Gal}(\overline{\mathbb{K}}/K)}.$$
The following theorem, of which expression is slightly changed, is given in [2], Chapter VII, Section 14, (14.1) Theorem.

**Theorem 5.** Let \( H^2(X, \mathbb{Q}) \) be the de Rham cohomology group of degree 2 with coefficients in \( \mathbb{Q} \). A compact complex manifold is a smooth projective variety over \( \mathbb{C} \) if and only if there exists a Hodge metric \( \omega_0 \) on \( X \), i.e., there exists a Kähler metric \( \omega_0 \) such that the de Rham cohomology class \([\omega_0]\) \( \in H^2(X, \mathbb{Q}) \).

Let \( X \) be a smooth projective variety over \( \mathbb{C} \) with a Hodge metric \( \omega_0 \) (cf. Theorem 5). The following, of which expression is slightly changed, is given in [2], Chapter VI, Section 8.2, (8.5) Hodge decomposition theorem.

**Theorem 6.** Let \( H^k(X, \mathbb{C}) \) be the de Rham cohomology group of degree \( k \). Then
\[
H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X, \mathbb{C}),
\]
where \( H^{p,q}(X, \mathbb{C}) \) is the set of de Rham cohomology classes defined by \((p,q)\)-forms on \( X \).

Let \( 0 \leq p \leq n := \dim X \) be an integer. An analytic \( p \)-cycle is a finite formal \( \mathbb{Q} \)-linear combination \( \sum c_l \Gamma_l \) \((c_l \in \mathbb{Q})\) of irreducible \( p \)-dimensional complex analytic subvarieties \( \{\Gamma_l\} \).

**Definition 7.** Let \( Y \subset X \). The restrictions to \( Y \) of global analytic \( p \)-cycles \( \Gamma = \sum c_l \Gamma_l \) are \( \Gamma \cap Y := \sum c_l (\Gamma_l \cap Y) \).

**Definition 8.** Let \( Y \subset X \). Two restrictions \( \Delta_1, \Delta_2 \) to \( Y \) of global analytic \( p \)-cycles are equivalent if
\[
\int_{\Delta_1} \eta = \int_{\Delta_2} \eta.
\]
for any closed \((p,p)\)-form \( \eta \) on \( X \). Let \( C_p(Y) \) be the set of equivalence classes defined by the restrictions to \( Y \) of global analytic \( p \)-cycles.

**Remark 9.** The set of equivalence classes of analytic \( p \)-cycles on \( X \) is the set of de Rham cohomology classes of analytic \( p \)-cycles on \( X \). Thus the old and the new definitions of \( C_p(X) \) coincide.

A complexified analytic \( p \)-cycle is a formal \( \mathbb{C} \)-linear combination \( \sum c_l \Gamma_l \) \((c_l \in \mathbb{C})\) of irreducible \( p \)-dimensional complex analytic subvarieties \( \{\Gamma_l\} \).

**Definition 10.** Let \( Y \subset X \). The restrictions to \( Y \) of global complexified analytic \( p \)-cycles \( \Gamma = \sum c_l \Gamma_l \) are \( \Gamma \cap Y := \sum c_l (\Gamma_l \cap Y) \).
Definition 11. Let \( Y \subset X \). Two restrictions \( \Delta_1, \Delta_2 \) to \( Y \) of global complexified analytic \( p \)-cycles are equivalent if
\[
\int_{\Delta_1} \eta = \int_{\Delta_2} \eta.
\]
for any closed \((p,p)\)-form \( \eta \) on \( X \). Let \( C_p(Y) \otimes \mathbb{C} \) be the set of equivalence classes defined by the restrictions to \( Y \) of global complexified analytic \( p \)-cycles.

Let \( 1 \leq p \leq n-1 \) be an integer. Let \( B_r(x) \) denote a closed ball with boundary in some coordinate and \( \mathcal{W} \) be the set of such balls, that is, \( U \in \mathcal{W} \) if there exist \( z_1 \in X \) and a coordinate \( x_1 \) around \( z_1 \) such that \( U = \{||x_1|| \leq r_1\} \) for some \( r_1 > 0 \). Let
\[
\mathcal{W}|_{B_r(x)^0} := \{U \subset B_r(x)^0 \mid x \in U^0, U \in \mathcal{W}\}.
\]

Definition 12. Two restrictions \( \omega_1, \omega_2 \) to \( B_r(x)^0 \) of global harmonic forms are equivalent if
\[
\int_{B_r(x)^0} \omega_1 \wedge \eta = \int_{B_r(x)^0} \omega_2 \wedge \eta
\]
for any closed form \( \eta \) on \( X \). Let \( H^{*,*}(B_r(x)^0, \mathbb{C}) \) be the set of equivalence classes of the restrictions to \( B_r(x)^0 \) of global harmonic \((*,*)\)-forms.

Lemma 13. Let \( x \in X \). There exists a closed ball \( B_r(x) \) of center \( x \in X \) and sufficiently small radius \( r > 0 \) such that for any point \( z_0 \in B_r(x)^0 \) and for some coordinate of \( B_r(x)^0 \) there exist an \((n-p)\)-dimensional complex linear subspace \( M \) in \( B_r(x)^0 \) through the origin and a \( p \)-dimensional complex linear subspace \( L \) in \( B_r(x)^0 \) orthogonal to \( M \) through \( z_0 \) which extends to a global analytic \( p \)-cycle.

Proof. Embed \( X \subset \mathbb{P}^N (N >> 0) \) and consider a closed ball of center \( x \in X \subset \mathbb{P}^N \) and sufficiently small radius \( r > 0 \) in \( \mathbb{P}^N \). The intersection of the ball and \( X \) is a closed ball \( B_r(x) \) in \( X \) of center \( x \in X \) and radius \( r \). Consider hyperplanes \( H_1, \ldots, H_{n-p} \) through \( z_0 \) in \( \mathbb{P}^N \). Then since \( r > 0 \) is sufficiently small there exist such hyperplanes such that \( H_1 \cap \cdots \cap H_{n-p} \cap X \) is a global analytic \( p \)-cycle on \( X \) and does not go through \( x \) and such that \( (H_1 \cap \cdots \cap H_{n-p} \cap X) \cap B_r(x)^0 \) is a manifold. Take such global analytic \( p \)-cycle of the form \( H_1 \cap \cdots \cap H_{n-p} \cap X \) as \( L \). The remaining statement is easy. The assertion follows.

Lemma 14. \( C_{n-s}(B_r(x)^0) \otimes \mathbb{C} \subset H^{s,s}(B_r(x)^0, \mathbb{C}) \).

Proof. Extend a complexified \((n-s)\)-cycle \( \Gamma \) to \( X \) and there exists a global harmonic \((s,s)\)-form \( \Omega \) on \( X \) such that
\[
\int_{\Gamma} \Xi = \int_{X} \Omega \wedge \Xi
\]
for any closed \((n-s,n-s)\)-form \( \Xi \) on \( X \). Thus \([\Gamma]|_{B_r(x)^0} = [\Omega]|_{B_r(x)^0}\), and the assertion follows.
**Definition 15.** Let $V, W$ and $E$ be vector spaces. A bilinear map $\Phi : V \times W \to E$ is nondegenerate if

$$\Phi(v, w) = 0 \ (\forall w \in W) \Rightarrow v = 0$$

(31)

and

$$\Phi(v, w) = 0 \ (\forall v \in V) \Rightarrow w = 0.$$  

(32)

Let $[\Gamma] \in C_p(B_r(x)^o) \otimes \mathbb{C}$ and $[\eta] \in H^{p,p}(B_r(x)^o, \mathbb{C})$. Let

$$\int_{\Gamma} [\eta] := \{(U, \frac{1}{|U|} \int_U [\Gamma]|u \wedge [\eta]|u)\}_{u \in \mathcal{W}|_{B_r(x)^o}}.$$  

(33)

**Lemma 16.** There exists a closed ball $B_r(x)$ of center $x \in X$ and sufficiently small radius $r > 0$ such that the map

$$C_p(B_r(x)^o) \otimes \mathbb{C} \times H^{p,p}(B_r(x)^o, \mathbb{C}) \to \text{Map}(\mathcal{W}|_{B_r(x)^o}, \mathbb{C})$$

(34)

$$([\Gamma], [\eta]) \mapsto \int_{\Gamma} [\eta],$$

(35)

is a nondegenerate bilinear map. Further it suffices to consider $[\Gamma]$ defined by the restrictions of global analytic $p$-cycles of which components are smooth on $B_r(x)^o$.

**Proof.** By Lemma 13 there exists a closed ball $B_r(x)$ of center $x \in X$ and sufficiently small radius $r > 0$ such that for any point $z_0 \in B_r(x)^o$ and for some coordinate of $B_r(x)^o$ there exist an $(n - p)$-dimensional complex linear subspace $M$ in $B_r(x)^o$ through the origin and a $p$-dimensional complex linear subspace $L$ in $B_r(x)^o$ orthogonal to $M$ through $z_0$ which extends to a global analytic $p$-cycle. Let $[\eta] \in H^{p,p}(B_r(x)^o, \mathbb{C})$. Let $z_0 \in B_r(x)^o$. Divide $\eta = \alpha_L + \beta$, where $\eta|_{z_0 + L(z_0)} = \alpha_L(z_0)$ for any $z_0$ near $z_0$. Assume $\alpha_L(z_0) \neq 0$ then it is obvious that

$$\int_{L \cap B_{|z_0| + \epsilon}(x)} \eta \neq 0 \ (\epsilon > 0 \text{ is small}).$$  

(36)

This contradiction shows $\alpha_L(z_0) = 0$ and $\eta(z_0) = \beta(z_0)$. Change coordinates and consider all such $L$ (cf. the proof of Lemma 13). Combining the resulting formulas it is, by an elementary argument of exterior algebra, obtained that $\eta(z_0) = 0$. Since $z_0 \in B_r(x)^o$ is arbitrary it follows that $[\eta] = 0$. Now it is proved that

$$\int_{\Gamma} [\eta] = \{(U, \frac{1}{|U|} \int_U [\Gamma]|u \wedge [\eta]|u)\}_{u \in \mathcal{W}|_{B_r(x)^o}} = \{(U, 0)\}_{u \in \mathcal{W}|_{B_r(x)^o}}$$

(37)

$$ (\forall [\Gamma] \in C_p(B_r(x)^o) \otimes \mathbb{C})$$

$$\Rightarrow [\eta] = 0.$$  

(38)

(39)
By Lemma 14 $C_{n-p}(B_r(x)\circlearrowleft) \otimes \mathbb{C} \subset H^{p,p}(B_r(x)\circlearrowleft, \mathbb{C})$ and $C_p(B_r(x)\circlearrowleft) \otimes \mathbb{C} \subset H^{n-p,n-p}(B_r(x)\circlearrowleft, \mathbb{C})$. Thus reversing the roles and considering $[\Gamma]$ corresponding to the restriction to $B_r(x)\circlearrowleft$ of a global harmonic form as an element of $H^{n-p,n-p}(B_r(x)\circlearrowleft, \mathbb{C})$ it follows that

$$
\int_{\Gamma} [\Gamma] = \{(U, \frac{1}{|U|} \int_{\Omega} [\Gamma]|u \wedge [\Gamma]|u)\}_{u \in \mathcal{W}|_{B_r(x)\circlearrowleft}} = \{(U, 0)\}_{u \in \mathcal{W}|_{B_r(x)\circlearrowleft}} \quad (40)
$$

$$(\forall [\Gamma] \in C_{n-p}(B_r(x)\circlearrowleft) \otimes \mathbb{C}) \quad (41)
$$

$$
\Rightarrow [\Gamma] = 0. \quad (42)
$$

The above two show the map (34)-(35) is a nondegenerate bilinear map. Further it suffices to consider $[\Gamma]$ defined by the restrictions of global analytic $p$-cycles of which components are smooth on $B_r(x)\circlearrowleft$. The assertion follows. \[\square\]

The following, of which expression is slightly changed, is given in [2], Chapter VI, Section 3.3, (3.17) Hodge isomorphism theorem.

**Theorem 17.** Let $X$ be a smooth projective variety over $\mathbb{C}$ or, more generally, a compact complex manifold. Let $\mathcal{H}^k(X, \mathbb{C})$ be the set of harmonic $k$-forms on $X$. Then $\mathcal{H}^k(X, \mathbb{C})$ is finite dimensional and each de Rham cohomology class is uniquely represented by a harmonic form. In particular $H^k(X, \mathbb{C}) \simeq \mathcal{H}^k(X, \mathbb{C})$.

Let $\omega$ be a global harmonic $(n-p,n-p)$-form on $X$ such that $[\omega] \in H^{n-p,p-en-p}(X, \mathbb{C}) \cap H^{2(n-p)}(X, \mathbb{Q})$. Observe that the set of harmonic forms on $X$ and thus $C_\alpha(B_r(x)\circlearrowleft)$ and $H^{\ast\ast}(B_r(x)\circlearrowleft, \mathbb{C})$ are finite dimensional (see Theorem 6 and Theorem 17).

**Definition 18.** The tangent space of a $C^1$-manifold $\Delta$ at $x \in \Delta$ is denoted by $T_x \Delta$. Two finite formal $C$-combinations $\sum c_l \Gamma_l$, $\sum c'_{l'} \Gamma'_{l'}$ ($c_l, c'_{l'} \in \mathbb{C}$) of $C^1$-manifolds $\{\Gamma_l\}$, $\{\Gamma'_{l'}\}$ intersect transversally if $T_x \Gamma_l \nsubseteq T_x \Gamma'_{l'}$ and $T_x \Gamma_l \nsubseteq T_x \Gamma'_{l'}$ for any $l, l'$ and for any $x \in \Gamma_l \cap \Gamma'_{l'}$.

**Lemma 19.** There exist a finite cover $\{U_\lambda\}_{\lambda=1}^\Lambda$ consisting of closed balls with boundary and $[\Gamma_\lambda] \in C_p(U_\lambda) \otimes \mathbb{C}$ ($1 \leq \lambda \leq \Lambda$) defined by the restriction of a global analytic $p$-cycle of which components are smooth on $U_\lambda$ satisfying the following: (i) for any global harmonic $(p,p)$-form $\eta$ on $X$

$$
\int_{\Gamma_\lambda} [\eta]|_{U_\lambda} = \int_{U_\lambda} [\omega]|_{U_\lambda} \wedge [\eta]|_{U_\lambda}, \quad (43)
$$

and (ii) if $U_\lambda \cap U_\mu \neq \emptyset$ then for any global harmonic $(p,p)$-form $\eta$ on $X$

$$
\int_{\Gamma_\lambda \cap U_\mu} [\eta]|_{U_\lambda \cap U_\mu} = \int_{\Gamma_\mu \cap U_\lambda} [\eta]|_{U_\lambda \cap U_\mu}. \quad (44)
$$

Furthermore the above are taken so that $\Gamma_\lambda$ and $\partial U_\lambda$ intersect transversally.
Proof. Since $C_p(B_r(x)^\circ) \otimes \mathbb{C} \subset H^{n-p,n-p}(B_r(x)^\circ, \mathbb{C})$ there exists a pairing

$$H^{n-p,n-p}(B_r(x)^\circ, \mathbb{C}) \times H^{p-p}(B_r(x)^\circ, \mathbb{C}) \to \text{Map}(\mathfrak{M}_{B_r(x)^\circ}, \mathbb{C})$$  (45)

extending the map (34)-(35). By Lemma 16 this new pairing is also nondegenerate. Observe that the old pairing is nondegenerate and $H^{n-p,n-p}(B_r(x)^\circ, \mathbb{C})$ and $H^{p-p}(B_r(x)^\circ, \mathbb{C})$ are finite dimensional. Thus it follows by linear algebra that $C_p(B_r(x)^\circ) \otimes \mathbb{C} = H^{n-p,n-p}(B_r(x)^\circ, \mathbb{C})$.

From above there exists $[\Gamma_x] \in C_p(B_r(x)^\circ)$ for each $x \in X$ such that for any $U \in \mathfrak{M}_{B_r(x)^\circ}$

$$\int_U [\Gamma_x]|_U \wedge [\eta]|_U = \int_U [\omega]|_U \wedge [\eta]|_U \quad (\forall [\eta] \in H^{p-p}(B_r(x)^\circ, \mathbb{C})).$$  (46)

In particular $[\Gamma_x]$ is such that

$$\int_{B_r(x)} [\Gamma_x]|_{B_r(x)} \wedge [\eta]|_{B_r(x)} = \int_{B_r(x)} [\omega]|_{B_r(x)} \wedge [\eta]|_{B_r(x)}$$

$$\quad (\forall [\eta] \in H^{p-p}(B_r(x)^\circ, \mathbb{C}) \ (0 < r' << r)).$$  (47)

Further $[\Gamma_x]$ is taken to be an equivalence class defined by the restriction of a global analytic $p$-cycle of which components are smooth on $B_r(x)^\circ$. Thus since $X$ is compact there exist a finite cover $\{U_\lambda\}_{\lambda=1}^\Lambda$ consisting of closed balls with boundary and $[\Gamma_\lambda] \in C_p(U_\lambda) \otimes \mathbb{C} \ (1 \leq \lambda \leq \Lambda)$ defined by the restriction of a global analytic $p$-cycle of which components are smooth on $U_\lambda$ satisfying the following: (i) for any global harmonic $(p,p)$-form $\eta$ on $X$

$$\int_{\Gamma_\lambda} [\eta]|_{U_\lambda} = \int_{U_\lambda} [\omega]|_{U_\lambda} \wedge [\eta]|_{U_\lambda}$$  (50)

and (ii) if $U_\lambda \cap U_\mu \neq \emptyset$ then for any global harmonic $(p,p)$-form $\eta$ on $X$

$$\int_{\Gamma_\lambda \cap U_\mu} [\eta]|_{U_\lambda \cap U_\mu} = \int_{\Gamma_\mu \cap U_\lambda} [\eta]|_{U_\lambda \cap U_\mu}.$$  (51)

Since $\Gamma_\lambda$’s are smooth, by shrinking $U_\lambda$’s, the above are taken so that $\Gamma_\lambda$ and $\partial U_\lambda$ intersect transversally. The assertion follows. \qed

{\{[\Gamma_\lambda]\}_{\lambda=1}^\Lambda} defines a de Rham cohomology class $[\omega]$.

**Lemma 20.** There exists an equivalence class defined by the restriction $\Gamma$ of a global complexified analytic $p$-cycle of which components are smooth on $U_1 \cup \cdots \cup U_\lambda$ such that $([\Gamma] - [\omega])|_{U_1 \cup \cdots \cup U_\lambda} = 0$ and that $\Gamma$ intersects with $\partial(U_1 \cup \cdots \cup U_\lambda)$ transversally.
Proof. We prove the assertion by induction on \( \lambda \).

When \( \lambda = 1 \). Define \( \Gamma := \Gamma_1 \) on \( U_1 \) and then \( [\Gamma]|_{U_1} = [\Gamma_1] = [\omega]|_{U_1} \). We note that \( \Gamma_1 \) is taken so that each component of \( \Gamma_1 \) is smooth on \( U_1 \) and that \( \Gamma \) intersects with \( \partial U_1 \) transversally. The assertion follows.

Assume for \( \lambda - 1 \) the assertion holds. There exists an equivalence class defined by the restriction \( \Delta \) of a global complexified analytic \( p \)-cycle which components are smooth on \( U_1 \cup \cdots \cup U_{\lambda-1} \) such that \( ([\Delta] - [\omega])|_{U_1 \cup \cdots \cup U_{\lambda-1}} = 0 \) and that \( \Delta \) intersects with \( \partial(U_1 \cup \cdots \cup U_{\lambda-1}) \) transversally. The restrictions \( \Delta, \Gamma_\lambda \) of global complexified analytic \( p \)-cycles correspond to those \( \Psi, \Phi_\lambda \) of global harmonic \( (n - p, n - p) \)-forms. By construction \( ([\Psi] - [\omega])|_{U_1 \cup \cdots \cup U_{\lambda-1}} \cap U_\lambda = ([\Delta] - [\omega])|_{U_1 \cup \cdots \cup U_{\lambda-1}} \cap U_\lambda = 0 \). It follows that \( \Psi|_{U_1 \cup \cdots \cup U_{\lambda-1}} \cap U_\lambda - \Phi_\lambda|_{U_1 \cup \cdots \cup U_{\lambda-1}} \cap U_\lambda = 0 \). Define

\[
\Phi := \begin{cases} 
\Psi \text{ (on } U_1 \cup \cdots \cup U_{\lambda-1}) \\
\Phi_\lambda + (\Psi - \Phi_\lambda) \text{ (on } (U_1 \cup \cdots \cup U_{\lambda-1}) \cap U_\lambda) \\
\Phi_\lambda \text{ (on } U_\lambda \setminus U_1 \cup \cdots \cup U_{\lambda-1})
\end{cases}
\]

By the inductive assumption RHS is well-defined. Let \( \gamma \) be a complex analytic variety appearing in \( \Delta \cap ((U_1 \cup \cdots \cup U_{\lambda-1}) \cap U_\lambda) - \Gamma_\lambda \cap ((U_1 \cup \cdots \cup U_{\lambda-1}) \cap U_\lambda) \). Observe that

\[
\int_{\partial((U_1 \cup \cdots \cup U_{\lambda-1}) \cap U_\lambda)} ([\Psi]|_{U_1 \cup \cdots \cup U_{\lambda-1}} \cap U_\lambda - [\Phi_\lambda]|_{U_1 \cup \cdots \cup U_{\lambda-1}} \cap U_\lambda) \wedge \theta
\]

\[
= \int_{\partial((U_1 \cup \cdots \cup U_{\lambda-1}) \cap U_\lambda)} 0 \wedge \theta
\]

\[
= 0
\]

for any \((2p - 1)\)-form \( \theta \) and \( \gamma \cap \partial((U_1 \cup \cdots \cup U_{\lambda-1}) \cap U_\lambda) \) is of measure 0 on \( \partial((U_1 \cup \cdots \cup U_{\lambda-1}) \cap U_\lambda) \).

Observe that each component of \( \Delta \) (resp. \( \Gamma_\lambda \)) is smooth on \( U_1 \cup \cdots \cup U_{\lambda-1} \) (resp. \( U_\lambda \)). Recall that, for any \( \mu \), \( \Gamma_\mu \) is taken so that \( \Gamma_\mu \) intersects with \( \partial U_\mu \) transversally (see Lemma 19). Thus \( \Delta \) intersects with \( \partial(U_1 \cup \cdots \cup U_{\lambda-1}) \) transversally and \( \Gamma \lambda \) does with \( \partial U_\lambda \) transversally. If one component \( \delta \) of \( \Delta \) and one component \( \delta_\lambda \) of \( \Gamma_\lambda \) are such that \( T_x \delta \neq T_x \delta_\lambda \) for some \( x \in \delta \cap \delta_\lambda \cap \partial(U_1 \cup \cdots \cup U_{\lambda-1}) \cap U_\lambda \) then \( \dim(\delta \cap \delta_\lambda) < p \). On the other hand \( \delta \cap \partial(U_1 \cup \cdots \cup U_{\lambda-1}) \) and \( \delta_\lambda \cap \partial U_\lambda \) are empty or of real dimension \((2p - 1)\). Thus the above measure property shows a contradiction and the union of the tangent spaces (and the base spaces) of components of \( \Delta \) on \( \partial((U_1 \cup \cdots \cup U_{\lambda-1}) \cap U_\lambda) \) and that of those of components of \( \Gamma_\lambda \) on \( \partial((U_1 \cup \cdots \cup U_{\lambda-1}) \cap U_\lambda) \) coincide. Further, from this, the above measure property shows that the coefficients of \( \Delta \cap ((U_1 \cup \cdots \cup U_{\lambda-1}) \cap U_\lambda) \) and \( \Gamma \lambda \cap ((U_1 \cup \cdots \cup U_{\lambda-1}) \cap U_\lambda) \) coincide. Let

\[
\Gamma := \begin{cases} 
\Delta \text{ (on } U_1 \cup \cdots \cup U_{\lambda-1}) \\
\Gamma_\lambda \text{ (on } U_\lambda \setminus (U_1 \cup \cdots \cup U_{\lambda-1}))
\end{cases}
\]
\( \Gamma \) intersects with \((\partial(U_1 \cup \cdots \cup U_{\lambda-1})) \cap U_{\lambda}\) transversally so that any component of \( \Gamma \) is locally a graph of \( C^1 \)-functions that are holomorphic outside a proper smooth manifold. By taking limit it is obvious that the functions satisfy Cauchy-Riemann equations and hence any component of \( \Gamma \) is smooth complex analytic on \( U_1 \cup \cdots \cup U_{\lambda} \). We note that \( \Gamma \) intersects with \( \partial(U_1 \cup \cdots \cup U_{\lambda}) \) transversally. The assertion follows. \( \square \)

**Lemma 21.** There exists a natural nondegenerate pairing

\[
H^{2(n-p)}(X, \mathbb{Q}) \times H^{2p}(X, \mathbb{Q}) \to \mathbb{Q}.
\]  

(57)

**Proof.** \( X \) is a smooth projective variety over \( \mathbb{C} \) with the Hodge metric \( \omega_0 \) so that

\[
*1 \in H^{2n}(X, \mathbb{Q}),
\]

(58)

where \( * \) is the Hodge’s star operator with respect to \( \omega_0 \). Thus there exists a natural nondegenerate pairing

\[
H^{2(n-p)}(X, \mathbb{Q}) \times H^{2p}(X, \mathbb{Q}) \to \mathbb{Q}
\]

(59)

\[
([\Theta], [\Xi]) \mapsto <[\Theta], [\Xi]>_{\omega_0},
\]

(60)

where

\[
<[\Theta], [\Xi]>_{\omega_0} *1 = [\Theta] \cup [\Xi].
\]

(61)

The assertion follows. \( \square \)

**Proof of Theorem 1.** It is clear that

\[
C_p(X) \subset H^{n-p,n-p}(X, \mathbb{C}) \cap H^{2(n-p)}(X, \mathbb{Q}).
\]

(62)

Thus it suffices to show the other inclusion. Let \([\omega] \in H^{n-p,n-p}(X, \mathbb{C}) \cap H^{2(n-p)}(X, \mathbb{Q})\). \( X = U_1 \cup \cdots \cup U_\lambda \) and by Lemma 20 it is concluded that there exists a de Rham cohomology class \([\Gamma]\) (cf. Theorem 17) defined by a complexified analytic \( p \)-cycle such that \([\Gamma] - [\omega] = 0\) on \( X \). Thus

\[
[\Gamma] = [\omega] \in H^{n-p,n-p}(X, \mathbb{C}) \cap H^{2(n-p)}(X, \mathbb{Q}).
\]

(63)

Assume each \([\Gamma_i]\) corresponds to a \( p \)-dimensional irreducible complex analytic subvariety and that \([\{[\Gamma_i]\}]\) forms a basis of \( C_p(X) \). Express \([\Gamma]\) as

\[
[\Gamma] = \sum_i c_i [\Gamma_i] \ (c_i \in \mathbb{C}).
\]

(64)

Extend \([\{[\Gamma_i]\}]\) to a basis of \( H^{n-p,n-p}(X, \mathbb{C}) \cap H^{2(n-p)}(X, \mathbb{Q})\) and then by Lemma 21, formula (62) and formula (63) it is obtained by linear algebra that \( c_i \in \mathbb{Q} \) (\( \forall i \)). Thus \( \Gamma \) is in fact an analytic \( p \)-cycle on \( X \). The assertion follows. \( \square \)
3 Proof of Theorem 2

Let \( a, b \) be distinct prime numbers. Let \( K \supset F_a \) be a field of finite extension of \( F_a \). Let \( \mathbb{V} \) be a representation of \( \text{Gal}(\overline{Q}/Q_b) \) over \( Q_b \). Let \( X_0 \) be a smooth projective scheme over \( K \). Let \( X_0(K) \) be the set of \( K \)-rational points of \( X_0 \).

**Definition 22.** A reduced subscheme \( \mathcal{W} \) of \( X_0(K) \) is of dimension \( p \) at \( x \in \mathcal{W} \) if it is locally defined by \( K \)-coefficiential regular functions \( P_1, \ldots, P_{n-p} \) around \( x \) and, letting \( \mathbb{P} := (P_1, \ldots, P_{n-p}) \), the Jacobian matrix of \( \mathbb{P} \) at \( x \) is of rank \( n - p \). In this case we write \( \dim_x \mathcal{W} = p \). A reduced subscheme \( \mathcal{W} \) of \( X_0(K) \) is of dimension \( p \) if \( \sup \dim_x \mathcal{W} = p \).

Let \( Z_p(X_0(K)) \) be the set of formal \( \mathbb{V} \)-linear combinations of irreducible \( p \)-dimensional reduced subschemes of \( X_0(K) \). Let \( \overline{K} \) be the algebraic separable closure of \( K \) and \( X_0(\overline{K}) \) the set of \( \overline{K} \)-rational points of \( X_0 \). Let \( \mathfrak{O}_{\overline{K}} \) be the ring of integers of \( \overline{K} \). Let \( \mathfrak{a} \subset \mathfrak{O}_{\overline{K}} \) be a maximal ideal such that \( \mathfrak{a} \cap \mathbb{Z} = a\mathbb{Z} \).

There exists a natural surjective map from the quotient field of \( \mathfrak{O}_{\overline{K}}/a \) to \( \overline{K} \). For \( \sigma \in \text{Gal}(\overline{Q}/Q) \) such that \( \sigma(a) \subset a \) let \( \sigma \in \text{Gal}(K/F_a) \) be the induced element. Let

\[
I_\mathfrak{a} := \{ \sigma \in \text{Gal}(\overline{Q}/Q) \mid \sigma(a) \subset a \text{ and } \sigma = \text{id} \}.
\]

Let \( C^{I_\mathfrak{a}} := \overline{\mathbb{Q}}^{I_\mathfrak{a}} \). Let \( \iota : K \hookrightarrow \overline{\mathbb{Q}}^{I_\mathfrak{a}} \hookrightarrow \overline{Q} \). Let \( X_0(C) \) be the set of \( C \)-rational points of the reduced scheme induced from \( X_0(K) \) via \( \iota \). Let \( (X_0)_C \to X(C) \) be the resolution of singularity. Let \( Z_p((X_0)_C) \) be the set of formal \( \mathbb{V} \)-linear combinations of irreducible \( p \)-dimensional reduced subschemes of \( (X_0)_C \). By Theorem 1 it is easy to prove

\[
Z_p((X_0)_C) \to H^{n-p,n-p}_\text{ét}(X_0)_C, \mathbb{V})
\]

is surjective. Let \( C_p((X_0)_C) \) be the image of \( Z_p((X_0)_C) \) under this map. Let \( H^{n-p,n-p}_\text{ét}(X_0(\overline{K}), \mathbb{V}) \) be the image of the map

\[
H^{n-p,n-p}_\text{ét}(X_0)_C, \mathbb{V}) \to H^{n-p,n-p}_\text{ét}(X_0)_C, \mathbb{V}) \to H^{n-p,n-p}_\text{ét}(X_0(\overline{K}), \mathbb{V})
\]

Let \( \text{Gal}(\overline{K}/K) \) act on the set \( C_p((X_0)_C)^{I_\mathfrak{a}} \) naturally. This induces an action of \( \text{Gal}(\overline{K}/K) \) on \( H^{n-p,n-p}_\text{ét}(X_0(\overline{K}), \mathbb{V}) \) via the surjective map

\[
\begin{align*}
C_p((X_0)_C) & \quad\to \quad H^{n-p,n-p}_\text{ét}(X_0)_C, \mathbb{V}) \quad\to\quad H^{n-p,n-p}_\text{ét}(X_0(\overline{K}), \mathbb{V})^{I_\mathfrak{a}} \\
C_p((X_0)_C)^{I_\mathfrak{a}} & \quad\to\quad H^{n-p,n-p}_\text{ét}(X_0(\overline{K}), \mathbb{V}).
\end{align*}
\]

**Definition 23.** Let

\[
\mathcal{P}_K : \quad Z_p((X_0)_C) \quad\to\quad H^{n-p,n-p}_\text{ét}(X_0)_C, \mathbb{V}) \\
Z_p(X_0(K)) \quad\to\quad H^{n-p,n-p}_\text{ét}(X_0(\overline{K}), \mathbb{V})^{\text{Gal}(\overline{K}/K)}. \quad (69)
\]
Two elements $\Gamma_1, \Gamma_2$ of $Z_p(X_0(K))$ is numerically equivalent if

$$\mathcal{P}_K(\Gamma_1) \cup \mathcal{P}_K(\Delta) = \mathcal{P}_K(\Gamma_2) \cup \mathcal{P}_K(\Delta)$$

(70)

for any $\Delta \in Z_{n-p}(X_0(K))$. In this case we write $\Gamma_1 \sim \Gamma_2$.

**Definition 24.** Let

$$\mathcal{P}_{\overline{K}} : \quad Z_p((X_0)_C) \rightarrow H_{\text{et}}^{n-p,n-p}((X_0)_C, V) \quad \uparrow \quad H_{\text{et}}^{n-p,n-p}(X_0(\overline{K}), V).$$

(71)

Two elements $\Gamma_1, \Gamma_2$ of $Z_p(X_0(\overline{K}))$ is numerically equivalent if

$$\mathcal{P}_{\overline{K}}(\Gamma_1) \cup \mathcal{P}_{\overline{K}}(\Delta) = \mathcal{P}_{\overline{K}}(\Gamma_2) \cup \mathcal{P}_{\overline{K}}(\Delta)$$

(72)

for any $\Delta \in Z_{n-p}(X_0(\overline{K}))$. In this case we write $\Gamma_1 \sim \Gamma_2$.

**Lemma 25.** Let $\mathcal{V}$ be the induced reduced subscheme of $X_0(\mathbb{C})$ from a reduced subscheme of $X_0(K)$ via $\iota$. Then no $K$-rational point of $\mathcal{V}$ is in the singular locus of $X_0(\mathbb{C})$.

**Proof.** By assumption any point in the image of $X_0(K)$ is smooth. The set of $K$-rational points of $\mathcal{V}$ are contained in the set of such points and the assertion follows. \qed

**Proof of Theorem 2.** Let $H_{\text{et}}^{n-p,n-p}(X_0(\mathbb{C}), V)$ be the image of the map

$$H_{\text{et}}^{n-p,n-p}((X_0)_C, V) \rightarrow H_{\text{et}}((X_0)_C, V) \rightarrow H_{\text{et}}(X_0(\mathbb{C}), V).$$

(73)

Let $C_p(X_0(\mathbb{C}))$ be the image of the map

$$Z_p((X_0)_C) \rightarrow H_{\text{et}}^{n-p,n-p}((X_0)_C, V) \quad \uparrow \quad Z_p(X_0(\mathbb{C})) \rightarrow H_{\text{et}}^{n-p,n-p}(X_0(\mathbb{C}), V).$$

(74)

By Lemma 25 no component of an element of

$$(C_p(X_0(\mathbb{C}))^I_a)^{\text{Gal}(\overline{K}/K)}$$

(75)

intersects with the singular locus of $X_0(\mathbb{C})$. It is obtained that

$$(C_p(X_0(\mathbb{C}))^I_a)^{\text{Gal}(\overline{K}/K)} = (C_p((X_0)_C)^I_a)^{\text{Gal}(\overline{K}/K)}.$$  

(76)

From the definitions it is easy to prove

$$(Z_p(X_0(K))/\sim) = (Z_p(X_0(\overline{K}))/\sim)^{\text{Gal}(\overline{K}/K)}$$

(77)

$$= (C_p(X_0(\mathbb{C}))^I_a)^{\text{Gal}(\overline{K}/K)} = (C_p((X_0)_C)^I_a)^{\text{Gal}(\overline{K}/K)}$$

(78)
and

\[ H_{\text{et}}^{n-p,n-p}(X_0(\overline{K}), \mathbb{V})^{\text{Gal}(\overline{K}/K)} = (H_{\text{et}}^{n-p,n-p}((X_0)_C, \mathbb{V})^I_{\text{et}})^{\text{Gal}(\overline{K}/K)}. \]  

(79)

Thus the cycle map

\[
\begin{align*}
C_p((X_0)_C) & \rightarrow H_{\text{et}}^{n-p,n-p}((X_0)_C, \mathbb{V}) \\
\uparrow & \\
Z_p(X_0(K)) & \sim H_{\text{et}}^{n-p,n-p}(X_0(\overline{K}), \mathbb{V})^{\text{Gal}(\overline{K}/K)}
\end{align*}
\]

(80)

is surjective and the assertion follows.

\[ \square \]

4 Appendix

Let \( K \supset \mathbb{Q} \) be a field of finite extension of \( \mathbb{Q} \). Let \( \mathbb{V} \) be a representation of \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) over \( \mathbb{Q} \). Let \( X_0 \) be a smooth projective scheme over \( K \). Let \( X_0(K) \) be the set of \( K \)-rational points of \( X_0 \).

**Definition 26.** A reduced subscheme \( \mathcal{W} \) of \( X_0(K) \) is of dimension \( p \) at \( x \in \mathcal{W} \) if it is locally defined by \( K \)-coefficiential regular functions \( P_1, \ldots, P_{n-p} \) around \( x \) and, letting \( \mathbb{P} := (P_1, \ldots, P_{n-p}) \), the Jacobian matrix of \( \mathbb{P} \) at \( x \) is of rank \( n - p \). In this case we write \( \dim_x \mathcal{W} = p \). A reduced subscheme \( \mathcal{W} \) of \( X_0(K) \) is of dimension \( p \) if \( \sup \dim_x \mathcal{W} = p \).

Let \( Z_p(X_0(K)) \) be the set of formal \( \mathbb{V} \)-linear combinations of irreducible \( p \)-dimensional reduced subschemes of \( X_0(K) \). Let \( X_0(\overline{\mathbb{Q}}) \) be the set of \( \overline{\mathbb{Q}} \)-rational points of \( X_0 \). Let \( \iota : K \rightarrow \overline{\mathbb{Q}} \). Let \( X_0(\mathbb{C}) \) be the set of \( \mathbb{C} \)-rational points of the reduced scheme induced from \( X_0(K) \) via \( \iota \). Let \( (X_0)_C \rightarrow X(\mathbb{C}) \) be the resolution of singularity. Let \( Z_p((X_0)_C) \) be the set of formal \( \mathbb{V} \)-linear combinations of irreducible \( p \)-dimensional reduced subschemes of \( (X_0)_C \). By Theorem 1 it is easy to prove

\[ Z_p((X_0)_C) \rightarrow H_{\text{et}}^{n-p,n-p}((X_0)_C, \mathbb{V}) \]  

(81)

is surjective. Let \( C_p((X_0)_C) \) be the image of \( Z_p((X_0)_C) \) under this map. Let \( H_{\text{et}}^{n-p,n-p}(X_0(\overline{\mathbb{Q}}), \mathbb{V}) \) be the image of the map

\[ H_{\text{et}}^{n-p,n-p}((X_0)_C, \mathbb{V}) \rightarrow H_{\text{et}}((X_0)_C, \mathbb{V}) \rightarrow H_{\text{et}}(X_0(\overline{\mathbb{Q}}), \mathbb{V}). \]  

(82)

Let \( \text{Gal}(\mathbb{C}/K) \) act on the set \( C_p((X_0)_C) \) naturally. This induces an action of \( \text{Gal}(\overline{\mathbb{Q}}/K) \) on \( H_{\text{et}}^{n-p,n-p}(X_0(\overline{\mathbb{Q}}), \mathbb{V}) \) via the surjective map

\[
\begin{align*}
C_p((X_0)_C) & \rightarrow H_{\text{et}}^{n-p,n-p}((X_0)_C, \mathbb{V}) \\
\downarrow & \\
H_{\text{et}}^{n-p,n-p}(X_0(\overline{\mathbb{Q}}), \mathbb{V}).
\end{align*}
\]

(83)
Definition 27. Let
\[
P_K : \begin{array}{l}
Z_p((X_0)_{\mathbb{C}}) \rightarrow H_{et}^{n-p,n-p}((X_0)_{\mathbb{C}}, \mathbb{V})
\end{array}
\]
\[
Z_p(X_0(K)) \rightarrow H_{et}^{n-p,n-p}(X_0(\overline{\mathbb{Q}}), \mathbb{V})^{Gal(\overline{\mathbb{Q}}/K)}.
\]
(84)

Two elements \( \Gamma_1, \Gamma_2 \) of \( Z_p(X_0(K)) \) is numerically equivalent if
\[
P_K(\Gamma_1) \cup P_K(\Delta) = P_K(\Gamma_2) \cup P_K(\Delta)
\]
(85)
for any \( \Delta \in Z_{n-p}(X_0(K)) \). In this case we write \( \Gamma_1 \sim \Gamma_2 \).

Definition 28. Let
\[
P_{\overline{\mathbb{Q}}} : \begin{array}{l}
Z_p((X_0)_{\mathbb{C}}) \rightarrow H_{et}^{n-p,n-p}((X_0)_{\mathbb{C}}, \mathbb{V})
\end{array}
\]
\[
Z_p(X_0(\overline{\mathbb{Q}})) \rightarrow H_{et}^{n-p,n-p}(X_0(\overline{\mathbb{Q}}), \mathbb{V}).
\]
(86)

Two elements \( \Gamma_1, \Gamma_2 \) of \( Z_p(X_0(\overline{\mathbb{Q}})) \) is numerically equivalent if
\[
P_{\overline{\mathbb{Q}}}(\Gamma_1) \cup P_{\overline{\mathbb{Q}}}(\Delta) = P_{\overline{\mathbb{Q}}}(\Gamma_2) \cup P_{\overline{\mathbb{Q}}}(\Delta)
\]
(87)
for any \( \Delta \in Z_{n-p}(X_0(\overline{\mathbb{Q}})) \). In this case we write \( \Gamma_1 \sim \Gamma_2 \).

Lemma 29. Let \( \mathcal{V} \) be the induced reduced subscheme of \( X_0(\mathbb{C}) \) from a reduced subscheme of \( X_0(K) \) via \( \iota \). Then no \( K \)-rational point of \( \mathcal{V} \) is in the singular locus of \( X_0(\mathbb{C}) \).

Proof. By assumption any point in the image of \( X_0(K) \) is smooth. The set of \( K \)-rational points of \( \mathcal{V} \) are contained in the set of such points and the assertion follows. \( \square \)

Theorem 30 (Tate Conjecture). The cycle map
\[
C_p((X_0)_{\mathbb{C}}) \rightarrow H_{et}^{n-p,n-p}((X_0)_{\mathbb{C}}, \mathbb{V})
\]
\[
Z_p(X_0(K))/\sim \rightarrow H_{et}^{n-p,n-p}(X_0(\overline{\mathbb{Q}}), \mathbb{V})^{Gal(\overline{\mathbb{Q}}/K)}
\]
(88)
is surjective.

Proof. Let \( H_{et}^{n-p,n-p}(X_0(\mathbb{C}), \mathbb{V}) \) be the image of the map
\[
H_{et}^{n-p,n-p}((X_0)_{\mathbb{C}}, \mathbb{V}) \rightarrow H_{et}((X_0)_{\mathbb{C}}, \mathbb{V}) \rightarrow H_{et}(X_0(\mathbb{C}), \mathbb{V}).
\]
(89)
Let \( C_p(X_0(\mathbb{C})) \) be the image of the map
\[
Z_p((X_0)_{\mathbb{C}}) \rightarrow H_{et}^{n-p,n-p}((X_0)_{\mathbb{C}}, \mathbb{V})
\]
\[
Z_p(X_0(\mathbb{C})) \rightarrow H_{et}^{n-p,n-p}(X_0(\mathbb{C}), \mathbb{V}).
\]
(90)
By Lemma 29 no component of an element of
\[ C_p(X_0(\mathbb{C}))^\text{Gal}(\mathbb{C}/K) \] intersects with the singular locus of \( X_0(\mathbb{C}) \). It is obtained that
\[ C_p(X_0(\mathbb{C}))^\text{Gal}(\mathbb{C}/K) = C_p((X_0)_\mathbb{C})^\text{Gal}(\mathbb{C}/K). \] (92)

From the definitions it is easy to prove
\[
(Z_p(X_0(K))/\sim) = (Z_p(X_0(\overline{\mathbb{Q}}))/\sim)^{\text{Gal}(\overline{\mathbb{Q}}/K)} \\
= C_p(X_0(\mathbb{C}))^{\text{Gal}(\mathbb{C}/K)} = C_p((X_0)_\mathbb{C})^{\text{Gal}(\mathbb{C}/K)}
\] (93)

and
\[
H^{n-p,n-p}_\text{et}(X_0(\overline{\mathbb{Q}}), \mathcal{V})^{\text{Gal}(\overline{\mathbb{Q}}/K)} = (H^{n-p,n-p}_\text{et}((X_0)_\mathbb{C}, \mathcal{V})^{\text{Gal}(\mathbb{C}/\overline{\mathbb{Q}})})^{\text{Gal}(\overline{\mathbb{Q}}/K)} \\
= H^{n-p,n-p}_\text{et}((X_0)_\mathbb{C}, \mathcal{V})^{\text{Gal}(\mathbb{C}/K)}.
\] (95)

Thus the cycle map
\[
C_p((X_0)_\mathbb{C}) \rightarrow H^{n-p,n-p}_\text{et}((X_0)_\mathbb{C}, \mathcal{V}) \\
Z_p(X_0(K))/\sim \downarrow \quad \downarrow \\
H^{n-p,n-p}_\text{et}(X_0(\overline{\mathbb{Q}}), \mathcal{V})^{\text{Gal}(\overline{\mathbb{Q}}/K)}
\] (97)

is surjective and the assertion follows.

\[ \square \]

References

[1] Pierre Deligne, The Hodge Conjecture
claymath.org/sites/default/files/hodge.pdf [Accessed: 18th January 2017]

[Accessed: 14th February 2016]


[5] Burt Totaro, Recent Progress on the Tate Conjecture
http://dx.doi.org/10.1090/bull/1588 [Accessed: 29th December 2017]