A solution to the smooth Poincaré conjecture

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Abstract

We construct a compact differential manifold that is homeomorphic
but not diffeomorphic to $S^4$. The problem is well-known as the smooth
Poincaré conjecture. Our proof is a new approach. A new method of the
introduction of a differential structure is used.

Keywords: Smooth Poincaré conjecture, Exotic $\mathbb{R}^4$, Embedding, Ordinary differential equations

1 Introduction

Poincaré conjecture was a long-standing problem. Smale solved when the di-

mension is greater than 4, Freedman when the dimension is 4 and Perelman
when the dimension is 3. Before these great works Milnor discovered a differential
manifold that is homeomorphic but not diffeomorphic to a canonical sphere
(exotic sphere). See [8]. Thanks to the many works thereafter the numbers of exotic
spheres of dimension ≤ 64 are known except for the case of dimension 4 (see [1]), in fact the existence of a 4-dimensional exotic sphere is not known
(the smooth Poincaré conjecture) and our main theorem gives an example.

Let $S^4$ be the canonical 4-sphere. We prove the following theorem.

Theorem 1. There exists a compact differential manifold that is homeomorphic
but not diffeomorphic to $S^4$.

It is known that there exists a manifold that is homeomorphic
but not diffeomorphic to $\mathbb{R}^4$ (exotic $\mathbb{R}^4$). See [3], Chapter 9, section 4, Theorem
9.4.3. The proof proceeds as follows: Let $\infty \in S^4$. Let $D_1 \ni \infty$ be a small disc
and $D_2 := S^4 \setminus \{\infty\}$. We introduce to $D_2$ the differential structure of an exotic
$\mathbb{R}^4$ and to $D_1 \setminus \{\infty\}$ the induced structure. We obtain a topological manifold
$M$ that is homeomorphic to $S^4$ and has an exotic $\mathbb{R}^4$ as a subset. We embed $M$
into $\mathbb{R}^N$ for some large $N$. Then we deform $\mathbb{R}^N$ so that $M \setminus \{\infty\}$ is embedded
into $\mathbb{R}^N \setminus \{0\}$ as a smooth manifold. Considering a set of carefully selected flows and deforming $M$ with the use of them we finally obtain a compact differential manifold that is homeomorphic but not diffeomorphic to $S^4$.

2 Proof of Theorem 1

We prove the main theorem. We break the proof into several steps.

Step 1 (Construction of $M$):

We first construct a (singular) manifold $M$ that has an exotic $\mathbb{R}^4$ as a subset.

Let $S^4$ be the canonical 4-sphere and $\infty \in S^4$. Let $D_1 \ni \infty$ be a small disc and $D_2 := S^4 \setminus \{\infty\}$. Since $S^4$ is an orientable differential manifold there exists an orientation-preserving diffeomorphism $\psi : D_1 \cap D_2 \to D_1 \cap D_2$. We introduce to $D_2$ the differential structure $\{(V_\alpha, \tau_\alpha)\}$ of an exotic $\mathbb{R}^4$ and to $D_1 \setminus \{\infty\}$ the differential structure $\{(\psi^{-1}(V_\alpha \cap D_1), \tau_\alpha \circ \psi)\}$. Then $\psi$ is an orientation-preserving diffeomorphism on $D_1 \cap D_2$ with respect to the new differential structures. Thus we obtain an orientable topological manifold $M$ such that $M \setminus \{\infty\}$ is an orientable differential manifold.

Step 2 (Embedding of $M$):

Next we embed $M$ into $\mathbb{R}^N$ for large $N \in \mathbb{N}$. This part is known (see [4], Chapter 1, Section 3, 3.4 Theorem).

Take a coordinate neighbourhood $(\infty \in U'_0 \subset \subset U_0)$ and introduce to $U_0 \setminus \{\infty\}$ the induced differential structure from that of $M \setminus \{\infty\}$. There exists a continuous function $\tilde{\rho}_0$ on $M$ such that

\begin{align*}
\tilde{\rho}_0 &= 1 \text{ on } U'_0, \quad (1) \\
0 &\leq \tilde{\rho}_0 \leq 1 \text{ on } U_0 \\
\text{and} \quad & (2) \\
\tilde{\rho}_0 &= 0 \text{ on } M \setminus U_0. \quad (4)
\end{align*}

Take a partition of unity $\{(\tilde{\rho}_\alpha, V_\alpha)\}$ on $D_2$ subordinate to $\{(V_\alpha, \tau_\alpha)\}$. Then since

\begin{equation}
M = (\text{supp} \tilde{\rho}_0) \circ \bigcup_{\alpha} (\text{supp} \tilde{\rho}_\alpha) \circ \quad (5)
\end{equation}

and since $M$ is compact there exist finitely many $\beta_j \ (j = 1, \ldots, s)$ such that

\begin{equation}
M = (\text{supp} \tilde{\rho}_0) \circ \bigcup_{j=1}^{s} (\text{supp} \tilde{\rho}_{\beta_j}) \circ \quad (6)
\end{equation}
Let
\[ \rho_{\beta_0} = \frac{\hat{\rho}_0}{\hat{\rho}_0 + \sum_{j=1}^{s} \hat{\rho}_{\beta_j}} \]  \hspace{1cm} (7)
and
\[ \rho_{\beta_j} = \frac{\hat{\rho}_{\beta_j}}{\hat{\rho}_0 + \sum_{j=1}^{s} \hat{\rho}_{\beta_j}} \ (j = 1, \ldots, s). \]  \hspace{1cm} (8)

\{\rho_{\beta_j}\}_{j=0}^{s} is a partition of unity of \( M \). Consider

\[ \Phi : M \ni p \mapsto (\rho_{\beta_0}(p)\tau_0(p), \ldots, \rho_{\beta_s}(p)\tau_s(p), \rho_{\beta_0}(p), \ldots, \rho_{\beta_s}(p)) \in \mathbb{R}^N \ (N \in \mathbb{N}), \]  \hspace{1cm} (9)

where \((U_0, \tau_0)\) is a local chart of \( M \). \( \Phi \) is a \( C^0 \)-embedding of \( M \) such that \( \Phi(M \setminus \{\infty\}) \) has an orientable \( C^\infty \)-structure.

We note that the \( C^0 \)-structure and the \( C^\infty \)-structure of \( \Phi(M \setminus \{\infty\}) \) may be taken so that they have the same orientation.

**Step 3 (Deformation of \( M \))**:

We deform \( \Phi(M) \) (and \( \mathbb{R}^N \)) by a homeomorphism \( T : \mathbb{R}^N \to \mathbb{R}^N \) so that \( T \circ \Phi(M \setminus \{\infty\}) \) is a smooth submanifold of \( \mathbb{R}^N \setminus \{0\} (= T(\mathbb{R}^N \setminus \{0\})) \).

Observe that any orientation-preserving homeomorphism of \( S^4 \) is isotopic to the identity (cf. [2]). Thus the orientation-preserving homeomorphism \( T'' \) of \( \Phi(M \setminus \{\infty\}) \) from the embedded structure to the orientable \( C^\infty \)-structure, which is extended to a homeomorphism \( T' \) of \( \Phi(M) \) that fixes 0, is extended to a homeomorphism \( T \) of \( \mathbb{R}^N \) by

\[ T(x, t) := S_{t|_{|x|}}(x) \ (x \in \Phi(M), 0 \leq |t| \leq \epsilon) \]  \hspace{1cm} (10)

in the tubular neighbourhood of \( \Phi(M) \), where \( S_0 \) is the isotopy from \( T' \) to the identity \((T = \text{id} \text{ outside the tubular neighbourhood})\). Then \( T \circ \Phi(\infty) = 0 \) and \( T \circ \Phi(M \setminus \{\infty\}) \) is an orientable smooth submanifold of \( \mathbb{R}^N \setminus \{0\} (= T(\mathbb{R}^N \setminus \{0\})) \).

Take as \( U_0 \) a sufficiently small neighbourhood of \( \infty \in M \). We deform \( T \circ \Phi(U_0) \) into \( \mathbb{R}^4 \times \{0\} \) by a flow. We note that \( \mathbb{R}^4 \times \{0\} \) is smooth.

Let \( f \) and \( g \) be the inverses of local charts of \( T \circ \Phi(U_0 \setminus \{\infty\}) \) and \( (\mathbb{R}^4 \times \{0\}) \setminus \)
\{0\}(\subset \mathbb{R}^N \setminus \{0\}) that extend as smooth functions (from \(\mathbb{R}^N \setminus \{0\}\) to \(\mathbb{R}^N \setminus \{0\}\)) to the whole of \(\mathbb{R}^N \setminus \{0\}\) so that \(f(x) = 0\) if and only if \(x = 0\) and \(g(x) = 0\) if and only if \(x = 0\). We note that such \(f\) and \(g\) does exist (consider \(C^\infty\)-coordinate change) and we may take a global map as \(g\). Consider the equation

\[
\begin{align*}
\frac{d}{dt}(\varphi_t) &= g - f, \\
\varphi_0(x) &= f(x). \tag{11}
\end{align*}
\]

Then by the theory of ordinary differential equations there exists a global solution \(\varphi_t\) on \(\mathbb{R}\). For the other local charts we perform the same procedure so that \(\Psi : \varphi_0 = f \mapsto \varphi_1 = g\) extends to a local diffeomorphism from \(T \circ \Phi(U_0 \setminus \{\infty\})\) to its image \(W_0\) in \((\mathbb{R}^4 \times \{0\}) \setminus \{0\}(\subset \mathbb{R}^N \setminus \{0\})\). Since \(U_0\) is expressed by a single \(C^0\)-coordinate \((\iota, V_0)\) there exists a homeomorphism \(\psi : V_0 \to V_0\) such that

\[
\begin{align*}
g^{-1} \circ \Psi \circ (T \circ \Phi) \circ \iota^{-1} \circ \psi|_{V_0 \setminus \{0\}} : \\
& z \mapsto \psi(z) \mapsto \iota^{-1}(\psi(z)) \\
& \mapsto f(f^{-1}(T \circ \Phi(\iota^{-1}(\psi(z)))))) \mapsto g(f^{-1}(T \circ \Phi(\iota^{-1}(\psi(z)))))) \\
& \mapsto f^{-1}(T \circ \Phi(\iota^{-1}(\psi(z)))) \tag{12 (13 (14 (15)}
\end{align*}
\]

is a homeomorphism from \(V_0 \setminus \{0\}\) to \(g^{-1}(W_0)\) (we note that \(f^{-1}\) is a chart chosen for each \(z \in V_0 \setminus \{0\}\) so that they are patched to a global homeomorphism). Thus \(\Psi\) is a diffeomorphism and extends to a homeomorphism from \(T \circ \Phi(U_0 \setminus \{\infty\}) \cup \{0\}\) to \(W_0 \cup \{0\}\). We note that \(W_0 \cup \{0\}\) has a \(C^\infty\)-structure.

Now \(W_0\) is diffeomorphic to \(T \circ \Phi(U_0 \setminus \{\infty\})\) (and thus to \(U_0 \setminus \{\infty\}\)) and \(W_0 \cup \{0\}\) is homeomorphic to \(T \circ \Phi(U_0 \setminus \{\infty\}) \cup \{0\}\) (and thus to \(U_0\)). We deform \(M\) by replacing \(U_0\) with \(W_0 \cup \{0\}\). Then the resulting manifold \(\hat{S}^4\) is smooth.

**Proof of Theorem 1.** \(\hat{S}^4\) is homeomorphic to \(M\), which is to \(S^4\). Thus \(\hat{S}^4\) and \(S^4\) are homeomorphic. On the other hand if \(\hat{S}^4\) and \(S^4\) are diffeomorphic then the map

\[
D_2 = \hat{S}^4 \setminus \{\infty\} \simeq S^4 \setminus \{\ast\} \simeq \mathbb{R}^4 \tag{16}
\]

is a diffeomorphism. This contradicts with the assumption that \(D_2\) is exotic. Thus \(\hat{S}^4\) and \(S^4\) are not diffeomorphic. \(\square\)

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**References**


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