A solution to smooth Poincaré conjecture

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Abstract

We construct a compact differential manifold that is homeomorphic but not diffeomorphic to $S^4$. The problem is well-known as smooth Poincaré conjecture. Our proof is a new approach. A new method of the introduction of a differential structure is used.

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1 Introduction

Poincaré conjecture was a long-standing problem. S. Smale solved when the dimension is greater than 4, M. H. Freedman when the dimension is 4 and G. Perelman when the dimension is 3. Before these great works J. W. Milnor discovered a differential manifold that is homeomorphic but not diffeomorphic to a canonical sphere (exotic sphere). See [4]. Thanks to the many works thereafter the numbers of exotic spheres of dimension $\leq 64$ are known except for the case of dimension 4 (see [1]), in fact the existence of a 4-dimensional exotic sphere is not known (smooth Poincaré conjecture) and our main theorem gives an example.

Let $S^4$ be the canonical 4-sphere. We prove the following theorem.

Theorem 1. There exists a compact differential manifold that is homeomorphic but not diffeomorphic to $S^4$.

It is known that there exists a differentiable manifold that is homeomorphic but not diffeomorphic to $\mathbb{R}^4$ (exotic $\mathbb{R}^4$). See [2], Chapter 9, section 4, Theorem 9.4.3. The proof proceeds as follows: Let $\infty \in S^4$. Let $D_1 \ni \infty$ be a small disc and $D_2 := S^4 \setminus \{\infty\}$. We introduce to $D_2$ the differential structure of an exotic $\mathbb{R}^4$ and to $D_1 \setminus \{\infty\}$ the induced structure. We obtain a topological manifold $M$ that is homeomorphic to $S^4$ and has an exotic $\mathbb{R}^4$ as a subset. We embed $M$
into $\mathbb{R}^N$ for some large $N$. Then we deform $\mathbb{R}^N$ so that $M \setminus \{\infty\}$ is embedded into $\mathbb{R}^N \setminus \{0\}$ as a smooth manifold. Considering a set of carefully selected flows and deforming $M$ with the use of them we finally obtain a compact differential manifold that is homeomorphic but not diffeomorphic to $S^4$.

2 Proof of Theorem 1

Lemma 2. There exists a (singular) manifold $M$ that has an exotic $\mathbb{R}^4$ as a subset.

Proof. Let $S^4$ be the canonical 4-sphere and $\infty \in S^4$. Let $D_1 \ni \infty$ be a small disc and $D_2 := S^4 \setminus \{\infty\}$. Since $S^4$ is an orientable differential manifold there exists an orientation-preserving diffeomorphism $\psi : D_1 \cap D_2 \to D_1 \cap D_2$. We introduce to $D_2$ the differential structure $\{(V_\alpha, \tau_\alpha)\}$ of an exotic $\mathbb{R}^4$ and to $D_1 \setminus \{\infty\}$ the differential structure $\{(\psi^{-1}(V_\alpha \cap D_1), \tau_\alpha \circ \psi)\}$. Then $\psi$ is an orientation-preserving diffeomorphism on $D_1 \cap D_2$ with respect to the new differential structures. Thus we obtain an orientable topological manifold $M$ such that $M \setminus \{\infty\}$ is an orientable differential manifold diffeomorphic to an exotic $\mathbb{R}^4$. The assertion follows. □

Lemma 3. For large $N \in \mathbb{N}$ there exists a $C^0$-embedding $\Phi : M \to \mathbb{R}^N$ such that $\Phi(M \setminus \{\infty\})$ has an orientable $C^\infty$-structure.

Remark 4. This part is known (see [3], Chapter 1, Section 3, 3.4 Theorem).

Proof. Take a coordinate neighbourhood $(\infty \in U'_0 \subset \subset U_0$ and introduce to $U_0 \setminus \{\infty\}$ the induced differential structure from that of $M \setminus \{\infty\}$. There exists a continuous function $\tilde{\rho}_0$ on $M$ such that

\[ \tilde{\rho}_0 = 1 \text{ on } U'_0, \]
\[ 0 \leq \tilde{\rho}_0 \leq 1 \text{ on } U_0 \]
\[ \text{and} \]
\[ \tilde{\rho}_0 = 0 \text{ on } M \setminus U_0. \]

Take a partition of unity $\{(\tilde{\rho}_\alpha, V_\alpha)\}$ on $D_2$ subordinate to $\{(V_\alpha, \tau_\alpha)\}$. Then since

\[ M = (\text{supp} \tilde{\rho}_0)^\circ \cup \bigcup_\alpha (\text{supp} \tilde{\rho}_\alpha)^\circ \]

and since $M$ is compact there exist finitely many $\beta_j (j = 1, \ldots, s)$ such that

\[ M = (\text{supp} \tilde{\rho}_0)^\circ \cup \bigcup_{j=1}^s (\text{supp} \tilde{\rho}_{\beta_j})^\circ. \]
Let
\[ \rho_{\beta_0} = \frac{\hat{\rho}_0}{\hat{\rho}_0 + \sum_{j=1}^s \hat{\rho}_{\beta_j}} \] (7)
and
\[ \rho_{\beta_j} = \frac{\hat{\rho}_{\beta_j}}{\hat{\rho}_0 + \sum_{j=1}^s \hat{\rho}_{\beta_j}} (j = 1, \ldots, s). \] (8)

\( \{\rho_{\beta_j}\}_{j=0}^s \) is a partition of unity of \( M \). Consider

\[ \Phi : M \ni p \mapsto (\rho_{\beta_0}(p)\tau_0(p), \ldots, \rho_{\beta_s}(p)\tau_s(p), \rho_{\beta_0}(p), \ldots, \rho_{\beta_s}(p)) \in \mathbb{R}^N \ (N \in \mathbb{N}), \] (9)

where \((U_0, \tau_0)\) is a local chart of \( M \). \( \Phi \) is a \( C^0 \)-embedding of \( M \) such that \( \Phi(M\setminus\{\infty\}) \) has an orientable \( C^\infty \)-structure \( \{\Phi(V_a), \tau_a \circ \Phi^{-1}\} \). The assertion follows. \( \Box \)

**Remark 5.** We note that the \( C^0 \)-structure and the \( C^\infty \)-structure of \( \Phi(M\setminus\{\infty\}) \) may be taken so that they have the same orientation.

**Definition 6.** \( \Phi(M) \) has a local tubular neighbourhood \( \varpi : O_2 \to \Phi(M) \) if for each \( \Phi(p) \in \Phi(M) \) there exists an open neighbourhood \( O_1 \subset \Phi(M) \) of \( \Phi(p) \) such that \( \varpi^{-1}(O_1) \) is a tubular neighbourhood of \( O_1 \).

**Lemma 7.** \( \Phi(M) \) has a local tubular neighbourhood.

**Proof.** For simplicity we show \( \Phi(M) \) has a local tubular neighbourhood on \( \Phi(\text{supp} \rho_{\beta_0})^\circ \). The proof of the other cases are the same. Let \( p \in (\text{supp} \rho_{\beta_0})^\circ \) and

\[ A'(p) := (\rho_{\beta_0}(p)\tau_0(p), \rho_{\beta_0}(p)), \] (10)
\[ A(p) := (\tau_0(p), \rho_{\beta_0}(p)), \] (11)
\[ B(p) := (\rho_{\beta_1}(p)\tau_1(p), \ldots, \rho_{\beta_s}(p)\tau_s(p), \rho_{\beta_1}(p), \ldots, \rho_{\beta_s}(p)). \] (12)

Let \( \eta \) be a locally defined homeomorphism around \( \Phi(p) \) given by

\[ \eta : \Phi(p) \mapsto (A(p), B(p)). \] (13)

We note that \( A'(p) \mapsto A(p) \) is a locally defined homeomorphism or \( (x_1, x_2) \mapsto (\frac{x_1}{x_2}, x_2) \) is a homeomorphism on \( \{x_2 \neq 0\} \). Let \( \theta \) be a locally defined homeomorphism around \( \eta \circ \Phi(p) \) given by

\[ \theta(y_1, y_2) = (\theta_1, \theta_2) := (y_1, y_2 - B \circ A^{-1}(y_1)). \] (14)

Then \( \Phi(M) \) is locally defined by \( \theta_2 \circ \eta = 0 \) and it has a local tubular neighbourhood at \( \Phi(p) \). The assertion follows. \( \Box \)
Lemma 8. \( \Phi(M) \) has a global tubular neighbourhood.

Proof. By Lemma 7, \( \Phi(M) \) has a local tubular neighbourhood \( O \) so that for any loop from \( b_0 \) in \( \Phi(M) \) there corresponds a germ of a \( C^0 \)-transition function of germs of local coordinates of \( \mathbb{R}^N \) at \( b_0 \). Note that a local coordinate at \( b_0 \) in \( \mathbb{R}^N \) is continued along the loop and we may take the same local coordinate at the endpoint for any loop of the same homotopy class. Thus the fundamental group of \( \Phi(M) \) acts on the germs of local coordinates of \( b_0 \) in \( \mathbb{R}^N \). \( \Phi(M) \) is homeomorphic to \( M \), which is to \( S^4 \), so the fundamental group is \( \{1\} \) and the action is trivial. This implies \( O \) contains a global tubular neighbourhood. \( \square \)

Lemma 9. \( \Phi(M) \) (and \( \mathbb{R}^N \)) is deformed by a homeomorphism \( T : \mathbb{R}^N \to \mathbb{R}^N \) so that \( T \circ \Phi(M \setminus \{ \infty \}) \) is a smooth submanifold of \( \mathbb{R}^N \setminus \{0\} \) (\( = T(\mathbb{R}^N \setminus \{0\}) \)).

Proof. Observe that as is well-known any orientation-preserving homeomorphism of \( S^4 \) is isotopic to the identity. Thus the orientation-preserving homeomorphism \( T' \) of \( \Phi(M \setminus \{ \infty \}) \) from the embedded structure to the orientable \( C^\infty \)-structure, which is extended to a homeomorphism \( T' \) of \( \Phi(M) \) that fixes 0, is extended to a homeomorphism \( T \) of \( \mathbb{R}^N \) by

\[
T(x, t) := S_{|t|/\epsilon}(x) \quad (x \in \Phi(M), 0 \leq |t| \leq \epsilon)
\]

in the tubular neighbourhood of \( \Phi(M) \), where \( S_0 \) is the isotopy from \( T' \) to the identity, and by

\[
T = \text{id} \quad \text{outside the tubular neighbourhood.}
\]

Then \( T \circ \Phi(\infty) = 0 \) and \( T \circ \Phi(M \setminus \{ \infty \}) \) is an orientable smooth submanifold of \( \mathbb{R}^N \setminus \{0\} \) (\( = T(\mathbb{R}^N \setminus \{0\}) \)). The assertion follows. \( \square \)

Lemma 10. Take as \( U_0 \) a sufficiently small neighbourhood of \( \infty \in M \). Then \( T \circ \Phi(U_0) \) is deformed into \( \mathbb{R}^4 \times \{0\} \) by a flow.

Remark 11. We note that \( \mathbb{R}^4 \times \{0\} \) is smooth.

Proof. Let \( f \) and \( g \) be local parameterizations of \( T \circ \Phi(U_0 \setminus \{ \infty \}) \) and \( (\mathbb{R}^4 \times \{0\}) \setminus \{0\} \) (\( \subset \mathbb{R}^N \setminus \{0\} \)) that extend as smooth functions (from \( \mathbb{R}^N \setminus \{0\} \) to \( \mathbb{R}^N \setminus \{0\} \)) to the whole of \( \mathbb{R}^N \setminus \{0\} \) so that \( f(x) = 0 \) if and only if \( x = 0 \) and \( g(x) = 0 \) if and only if \( x = 0 \). We note that such \( f \) and \( g \) does exist (consider \( C^\infty \)-coordinate change) and we may take a global map as \( g \). Consider the equation

\[
\begin{align*}
\frac{d}{dt}(\phi_t) &= g - f, \\
\phi_0(x) &= f(x).
\end{align*}
\]

Then by the theory of ordinary differential equations there exists a global solution \( \phi_t \) on \( \mathbb{R} \). For the other local parameterizations we perform the same procedure. Take \( f, g \) so that \( f \) is defined on a covering space \( \mathcal{U} \) of a domain in \( \mathbb{R}^4 \setminus \{0\} \) (or \( \mathbb{R}^4 \times \{0\} \setminus \{0\} \) and \( g \) is a diffeomorphism defined on \( \mathbb{R}^2 \setminus \{0\} \) (or \( \mathbb{R}^4 \times \{0\} \setminus \{0\} \). We note that \( f \) is a local homeomorphism from \( \mathcal{U} \) to
$T \circ \Phi(U_0 \setminus \{\infty\})$ and $T \circ \Phi(U_0 \setminus \{\infty\})$ is simply connected, thus $f$ is a homeomorphism. With these choices $\Psi : \varphi_0 = f \mapsto \varphi_1 = g$ extends to a local diffeomorphism from $T \circ \Phi(U_0 \setminus \{\infty\})$ to its image $W_0$ in $(\mathbb{R}^4 \times \{0\}) \setminus \{0\} \subset \mathbb{R}^N \setminus \{0\}$. Let $U_0$ be expressed by a single $C^0$-coordinate $(\iota, V_0)$. Then

\begin{align}
  g^{-1} \circ \Psi \circ (T \circ \Phi) \circ \iota^{-1}|_{V_0 \setminus \{0\}} : \\
  z \mapsto \iota^{-1}(z) \\
  \mapsto f(f^{-1}(T \circ \Phi(\iota^{-1}(z)))) \mapsto g(f^{-1}(T \circ \Phi(\iota^{-1}(z)))) \\
  \mapsto f^{-1}(T \circ \Phi(\iota^{-1}(z)))
\end{align}

is a multi-valued map from $V_0 \setminus \{0\}$ to $g^{-1}(W_0)$. Its inverse is a covering map from $g^{-1}(W_0)$ to $V_0 \setminus \{0\}$ and $V_0 \setminus \{0\}$ is simply connected. It follows that $g^{-1} \circ \Psi \circ (T \circ \Phi) \circ \iota^{-1}|_{V_0 \setminus \{0\}}$ is a homeomorphism. Thus $\Psi$ is a diffeomorphism and extends to a homeomorphism from $T \circ \Phi(U_0 \setminus \{\infty\}) \cup \{0\}$ to $W_0 \cup \{0\}$. The assertion follows.

**Remark 12.** We note that $W_0 \cup \{0\}$ has a $C^\infty$-structure.

**Lemma 13.** $M$ is deformed into a smooth manifold.

**Proof.** By Lemma 10, $W_0$ is diffeomorphic to $T \circ \Phi(U_0 \setminus \{\infty\})$ (and thus to $U_0 \setminus \{\infty\}$) and $W_0 \cup \{0\}$ is homeomorphic to $T \circ \Phi(U_0 \setminus \{\infty\}) \cup \{0\}$ (and thus to $U_0$). We deform $M$ by replacing $U_0$ with $W_0 \cup \{0\}$. Then the resulting manifold $\hat{S}^4$ is smooth. The assertion follows.

**Proof of Theorem 1.** $\hat{S}^4$ is homeomorphic to $M$, which is to $S^4$. Thus $\hat{S}^4$ and $S^4$ are homeomorphic. On the other hand if $\hat{S}^4$ and $S^4$ are diffeomorphic then the map

$$D_2 = \hat{S}^4 \setminus \{\infty\} \simeq S^4 \setminus \{\ast\} \simeq \mathbb{R}^4$$

is a diffeomorphism. This contradicts with the assumption that $D_2$ is exotic. Thus $S^4$ and $\hat{S}^4$ are not diffeomorphic. The assertion follows from this.

**Remark 14.** As for classifying smooth manifolds by the well-known invariants 4-dimensional (exotic) spheres are all in the same category and any great theory cannot apply. The importance of establishing a new concept is not well-known. Thus the proof of smooth Poincaré conjecture is a surprisingly new approach. In fact it has many consequences: Once we prove a property of nonlinear partial differential equations on an exotic sphere restricted to the sphere minus finitely many points we may extend the result to the cases of the sphere. (Add the points and resolve singularity.)
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