The basic theory of turbulence

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Abstract

The numerical simulation of turbulences is a central of the next generation of fluid dynamics. Let $I := [0, T]$ ($T > 0$) be an interval. We prove the existence, smoothness and uniqueness of solutions of Navier-Stokes equations on $I \times (\mathbb{R}^3/\mathbb{Z}^3)$ and on $I \times \mathbb{R}^3$. Then we develop a theory of artificial intelligence. By a generality of our artificial intelligence it is possible to determine how to deal with any flow with indeterminacy automatically if the algorithm is in the direction of our rubric. A nondeterministic Turing Machine is used.

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1 Introduction

We investigate the following Navier-Stokes equations:

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= -u \cdot \nabla u + \nu \Delta u - \nabla p + f, \\
\nabla \cdot u &= 0.
\end{aligned}
\]

The linear partial differential equations are rigorously examined (cf. [14]). As to Navier-Stokes equations the global existence of smooth solutions of Navier-Stokes equations is a well-known problem (see e.g. [19], [6]). The following theorem, of which expression is slightly changed, is proved in [12].

Theorem 1. Let $\nu > 0$. Let $f = 0$. Let $u_0 \in L^2(\mathbb{R}^3)$ such that $\nabla \cdot u_0 = 0$. Then there exists a weak solution $u$ satisfying equation (1), i.e. for any compactly supported $\phi \in C^2((0, \infty) \times \mathbb{R}^3)^3$ with $\nabla \cdot \phi = 0$

\[
\int_0^\infty \int_{\mathbb{R}^3} (u \cdot \frac{\partial \phi}{\partial t} + \sum_j \sum_k u_k u_j \partial_k \phi_j + u \cdot \nu \Delta \phi) dx \, dt = 0.
\]

\[\text{2}\]
holds. Further for $t \in (0, \infty)$

$$||u(t, \cdot)||_{L^2(\mathbb{R}^3)}^2 + \nu \int_0^t ||\nabla u(s, \cdot)||_{L^2(\mathbb{R}^3)}^2 ds \leq ||u_0||_{L^2(\mathbb{R}^3)}^2,$$

(3)

and

$$||u(t, \cdot) - u_0||_{L^2(\mathbb{R}^3)} \to 0 \ (t \to +0).$$

(4)


Let $\nu > 0$ and $I := [0, T] \ (T > 0)$. Let

$$W_n := C_\infty(I, C_\infty(\mathbb{R}^3/\mathbb{Z}^3)^n).$$

(5)

Let

$$(W_1)_\nabla := \{ v \mid v = \nabla w \ (\exists w \in W_1) \}.$$

(6)

We prove the following theorem.

**Theorem 2.** Let $\nu > 0$. Let $I := [0, T] \ (T > 0)$. Let $u_0 \in C_\infty(\mathbb{R}^3/\mathbb{Z}^3)^3$ such that $\nabla \cdot u_0 = 0$ and let $f \in W_3$. Then there exists an unique $(u, \nabla p) \in W_3 \times (W_1)_\nabla$ such that

$$\begin{cases}
\partial u / \partial t = -(u \cdot \nabla)u + \nu \Delta u - \nabla p + f, \\
\nabla \cdot u = 0, \\
uuut_{t=0} = u_0.
\end{cases}$$

(7)

The proof of Theorem 2 proceeds as follows.

Let

$$\Phi : W_3 \times (W_1)_\nabla \to W_3 \times (W_3)_{\text{div}} \times C_\infty(\mathbb{R}^3/\mathbb{Z}^3)^3$$

be given by

$$(u, P) \mapsto \begin{bmatrix}
\dot{u} + (u \cdot \nabla)u - \nu \Delta u + P \\
\nabla \cdot u \\
uuut_{u(0)}
\end{bmatrix},$$

(9)

where

$$(W_3)_{\text{div}} := \{ v \mid v = \nabla \cdot w \ (\exists w \in W_3) \}.$$ 

(10)

$\Phi$ is $C_\infty$ in $(u, P)$ and the map

$$d\Phi : (h, \beta) \mapsto \begin{bmatrix}
\dot{h} + (u \cdot \nabla)h + (h \cdot \nabla)u - \nu \Delta h + \beta \\
\nabla \cdot h \\
uuut_{h(0)}
\end{bmatrix}$$

(11)

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is a linear isomorphism from
\[ W_3 \times (W_1)_V \] (12)
to
\[ \{(a, b, c) \in W_3 \times (W_3)_{\text{div}} \times C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3 \mid \nabla \cdot c = b(0)\}. \] (13)

Let
\[ \mathcal{X} := W_3 \times (W_1)_V \] (14)
and
\[ \mathcal{Y} := \{r := (f, r_2, u_0) \in W_3 \times (W_3)_{\text{div}} \times C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3 \mid \nabla \cdot u_0 = r_2(0)\}. \] (15)

Let
\[ \mathcal{Y} := \Phi(\mathcal{X}). \] (16)

Then \( \Phi \) is a map from \( \mathcal{X} \) to \( \mathcal{Y} \) and \( \mathcal{Y} \) is a Fréchet manifold. Let \( p = (u, P) \in \mathcal{X} \) be an arbitrary point. Let
\[ \varphi_p : \mathcal{X} \xrightarrow{\cong} T_p \mathcal{X} \] (17)
and
\[ \psi_{\Phi(p)} : \mathcal{Y} \xrightarrow{\cong} T_{\Phi(p)} \mathcal{Y} \] (18)
be canonical isomorphisms. Let
\[ U_p := \varphi_p^{-1}(T_p \mathcal{X}) (= \mathcal{X}) \] (19)
and
\[ V_{\Phi(p)} := \psi_{\Phi(p)}^{-1}(T_{\Phi(p)} \mathcal{Y}). \] (20)

Let \( H^{k_0}(\mathbb{R}^3/\mathbb{Z}^3) \) be the Sobolev space and
\[ W^{k_1, k_2}_n := C^{k_1}(I, H^{k_2}(\mathbb{R}^3/\mathbb{Z}^3)^n). \] (21)

Let \( k \in \mathbb{N} \). Let
\[ (W_3^{k,k+2})_{\text{div}} := \{v \mid v = \nabla \cdot w \ (\exists w \in W_3^{k,k+2})\}. \] (22)

Introduce to \( \mathcal{Y} \) the relative topology as a subset of
\[ W_3^{k,k} \times (W_3^{k,k+2})_{\text{div}} \times H^{k+2}(\mathbb{R}^3/\mathbb{Z}^3)^3, \] (23)
which defines a seminorm $p_k$ (it is in fact a norm) of $\mathcal{Y}$. Introduce to $T_{\Phi(p)}\mathcal{Y}$ the topology induced from $(\psi_{\Phi(p)})^{-1}|_{T_{\Phi(p)}\mathcal{Y}}$, to $T_p\mathcal{X}$ the topology induced from $(\psi_{\Phi(p)})^{-1}|_{T_{\Phi(p)}\mathcal{X}} \circ d\Phi(p)$, to $U_p$ the topology induced from $(\psi_{\Phi(p)})^{-1}|_{T_{\Phi(p)}\mathcal{Y}} \circ d\Phi(p) \circ \varphi_p|_{U_p}$, and to $V_{\Phi(p)}$ the topology induced from $(\psi_{\Phi(p)})^{-1}|_{T_{\Phi(p)}\mathcal{Y}} \circ \psi_{\Phi(p)}|_{V_{\Phi(p)}}$.

From above $U_p = \mathcal{X}$ and $V_{\Phi(p)} = \mathcal{X}$. The above topologies are induced by a seminorm $p_k$ such that $p_k(r)$ implies $r = 0$ (for $r \in \mathcal{Y}$). Also $U_p$ and $V_{\Phi(p)}$ are equipped with the ordinary topologies as $\mathcal{X}$ and $\mathcal{X}$. Then $q \mapsto d\Phi(q)$ is continuous with respect to the ordinary topology of $U_p$ and the topology of $T_{\Phi(p)}V_{\Phi(p)}$ $=$ $T_{\Phi(p)}\mathcal{Y}$ induced by $p_k$.

Let $Y$ be a Banach manifold and $B$ a Banach space. Let $L(q) : T_q Y \to B$ ($q \in Y$) be a map. Then the set of accumulation points of the sequences

$$\sum_{l} L(\gamma)(\xi_l)j(\xi_l)|E_l|,$$

as $j \to \infty$, where $\gamma : [0, 1] \to Y$ runs over all smooth paths from $p \in Y$ to $q \in Y$, $\{E_l\}$ all sequences of measurable sets of $[0, 1]$ such that $\prod_{l} E_{l}^j = [0, 1]$ and $\sup_{l} E_{l}^j \to 0$ ($j \to \infty$) and $\xi_l$ all elements of $E_l$, if exists, is denoted by $\int_{p} L(q)$. The set $X$ equipped with the topology induced by $p_k$ is, if exists, denoted by $X^w$.

With the use of a result of [17] it is proved that there exist a sufficiently small convex neighbourhood $\mathcal{X}_p$ of $p \in U_p$ with respect to the ordinary topology and a sufficiently small neighbourhood of $0 \in (T_{\Phi(p)}\mathcal{Y})^w$ identified with a sufficiently small neighbourhood $\mathcal{Y}_{\Phi(p)}^w$ of $\Phi(p) \in \mathcal{Y}_{\Phi(p)}$ such that the multi-valued map $q \in \mathcal{X}_{p}^w \mapsto \int_{p} L(q') \in \mathcal{Y}_{\Phi(p)}^w$ has a continuous branch. $\mathcal{X}_{p}$ and $\mathcal{X}_{p}$ are identified with the corresponding neighbourhoods of $0 \in T_{\Phi} \mathcal{X}_p$ ($q \in \mathcal{X}_p$) and $T_{\Phi} \mathcal{X}_p$ ($Q \in \mathcal{X}_p$). The same argument holds for $d(d\Phi)$. It follows that $d\Phi : (\mathcal{X}_{p}^w)^w \to (T\mathcal{Y}_{\Phi(p)})^w$ is a continuous map and the Fréchet derivative of $\Phi : \mathcal{X}_{p} \to \mathcal{Y}_{\Phi(p)}$ is equal to $d\Phi$ on $(\mathcal{X}_{p}^w)^w$.

Similarly $d(d\Phi) : (\mathcal{X}_{p}^w)^w \to (TT\mathcal{Y}_{\Phi(p)})^w$ is a continuous map and the Fréchet derivative of $d\Phi : (\mathcal{X}_{p}^w)^w \to (T\mathcal{Y}_{\Phi(p)})^w$ is equal to $d(d\Phi)$ on $(\mathcal{X}_{p}^w)^w$. Let $L((T_p\mathcal{X}_p)^w, (T_p\mathcal{Y}_{\Phi(p)})^w)$ be the set of continuous linear operators from $(T_p\mathcal{X}_p)^w$ to $(T_p\mathcal{Y}_{\Phi(p)})^w$. Then there exists $M' > 0$ such that

$$\|d\Phi(q) - d\Phi(q')\|_{L((T_p\mathcal{X}_p)^w, (T_p\mathcal{Y}_{\Phi(p)})^w)} \leq M'\|q' - q\|_{\mathcal{X}_p^w},$$

for $q, q', q' - q \in \mathcal{X}_p^w$, and such that

$$\|\Phi(q') - \Phi(q) - d\Phi(q)(q' - q)\|_{\mathcal{Y}_{\Phi(p)}^w} \leq M'\|q' - q\|_2^2 \mathcal{X}_p^w,$$
for $q, q', q' - q \in \mathscr{U}_p^w$.

Take the completions $\mathcal{U}_p^k$ and $\mathcal{V}_p^k(\phi)$ of $\mathscr{U}_p^w$ and $\mathscr{V}_p^w$ induced from $p_k$. From above, shrinking $\mathcal{U}_p^k$ and $\mathcal{V}_p^k(\phi)$ if necessary, $d\Phi$ extends to a continuous map, $\Phi$ extends to a $C^2$-map from $\mathcal{U}_p^k$ to $\mathcal{V}_p^k(\phi)$ and the Fréchet derivative with respect to the topologies of $\mathcal{U}_p^k$ and $\mathcal{V}_p^k(\phi)$ of the extended $\Phi$ at $q$ is equal to the value $d\Phi(q)$ at $q$ of the extended $d\Phi$. $d\Phi(p)$ extends to a topological isomorphism between the completions $T_p\mathcal{U}_p^k$ and $T_p\mathcal{V}_p^k(\phi)$ of $(T_p\mathscr{X})^w$ and $(T_p\mathscr{Y})^w$ induced from $p_k$ so that since $\mathscr{X}_p$ is sufficiently small $d\Phi(q) (q \in \mathcal{U}_p^k)$ is a topological isomorphism. By Inverse Function Theorem there exist sufficiently small neighbourhoods $U_p$ and $V_{\phi(p)}$ of $p \in \mathcal{U}_p^k$ and $\Phi(p) \in \mathcal{V}_{\phi(p)}^k$ such that the extended

$$ \Phi : U_p \xrightarrow{\cong} V_{\phi(p)} \quad (27) $$

is an isomorphism. Note that it is clear that $V_{\phi(p)} (p \in \mathscr{X})$ form a Banach manifold.

Replacing $I$ with an interval $J := [t_0, t_0 + T_0]$ ($0 \leq t_0 < T$ and $T_0 > 0$ is small) we define $\mathscr{X}^J$, $\mathscr{Y}^J$, $U_p^J$, $\Phi^J$ etc. in the same way as $\mathscr{X}$, $\mathscr{Y}$, $U_p$, $\Phi$ etc. Then it is shown that $U_p^J (p \in \mathscr{X}^J)$ form a Banach manifold.

We shall prove the local existence and uniqueness of a sufficiently smooth solution of equation (7) (of which smoothness depends on $k \in \mathbb{N}$).

Let $t_0 \in I$. Introduce to the inductive limits $\mathcal{U}_{t_0}$ of $\bigcup_p U_p^J$ for $J \ni t_0$ and $\mathcal{C}_{t_0}$ of

$$ \mathcal{X}^{(k), J} := \{(f, r_2, u_0) \in (W_3^{k,k})^J \times \left( (W_3^{k,k+2})^J \right)_{\text{div}} \times H^{k+2}(\mathbb{R}^3/\mathbb{Z}^3)^3 \right) \quad (28) $$

$$ \left| \nabla \cdot u_0 = r_2(t_0) \right| \quad (29) $$

for $J \ni t_0$ the natural topologies (the quotient topologies induced from the maps $\prod_J U_p^J \to \mathcal{U}_{t_0}$ and $\prod_J \mathcal{X}^{(k), J} \to \mathcal{C}_{t_0}$). Then the induced map $\tilde{\Phi}_{t_0} : \mathcal{U}_{t_0} \to \mathcal{C}_{t_0}$ is a homeomorphism. We obtain a bijection $\tilde{\Phi} : \mathcal{U} := \prod_{t_0} \mathcal{U}_{t_0} \to \mathcal{C} := \prod_{t_0} \mathcal{C}_{t_0}$ and introducing a sheaf structure to $\mathcal{C}$ induced from $\mathcal{U}$ through this bijection a sheaf isomorphism, where the topology of $I$ is generated by $J$'s. This proves the local existence and uniqueness of equation (7).

Let $\Omega \subset \mathbb{R}^3/\mathbb{Z}^3$. Let

$$ W_{\Omega,n} := C^\infty(I, C^\infty(\Omega)^n) \quad (30) $$

$$ (W_{\Omega,1})^\nabla := \{ v \mid v = \nabla w \ (\exists w \in W_{\Omega,1}) \} \quad (31) $$

$$ \mathscr{X}_\Omega := W_{\Omega,3} \times (W_{\Omega,1})^\nabla \quad (32) $$
Let \((f, r_2, u_0) \in \mathcal{X}\). Let \(x \in \mathbb{R}^3/\mathbb{Z}^3\). Then there exist a compact neighbourhood \(K_x\) of \(x\) and \((u^{K_x}, \nabla p^{K_x}) \in \mathcal{X}_{K_x}\) satisfying equation (7) on \(I \times K_x\). Consider a family \(\{(u^{K_x}, \nabla p^{K_x})\}_x\). Since \(\mathbb{R}^3/\mathbb{Z}^3\) is compact we may obtain a finite set \(\{x_\lambda\}\) such that each \(K_{x_\lambda}\) intersects with another in a set of Lebesgue measure 0 and \(\bigcup_{\lambda} K_{x_\lambda} = \mathbb{R}^3/\mathbb{Z}^3\). It follows that there exists \((u, \nabla p) \in (L^2(I \times (\mathbb{R}^3/\mathbb{Z}^3))^3)^2\) that is smooth a.e. and satisfies equation (7) a.e. From above it is proved that \(\Phi : \mathcal{X} \to \mathcal{Y}\) is a homeomorphism. Extend \(\Phi^{-1}\) to a continuous map from \(\mathcal{Y}(\subset \mathcal{X})\) to

\[
\{ (u, \nabla p) \in (L^2(I \times (\mathbb{R}^3/\mathbb{Z}^3))^3)^2 \mid u, \nabla p \text{ are smooth a.e.} \}.
\] (33)

Then considering the convolutions with mollifiers it is proved that \((u, \nabla p) \in \Phi^{-1}(\mathcal{Y})\) and thus \((u, \nabla p) = \Phi^{-1}((f, r_2, u_0))\). Let \((s, y) \in I \times (\mathbb{R}^3/\mathbb{Z}^3)\) be a singular point of \((u, \nabla p)\). Take another finite decomposition \(K_y = \bigcup_{\mu} K_{x_\mu}\) in \(\mathbb{R}^3/\mathbb{Z}^3\) and construct \((u_1, \nabla p_1)\) that is smooth at \((s, y)\) and satisfies equation (7) a.e. Since \((u, \nabla p) = \Phi^{-1}((f, r_2, u_0)) = (u_1, \nabla p_1)\) it follows that \((u, \nabla p)\) is smooth at \((s, y)\). This contradiction shows that \((u, \nabla p) \in \mathcal{X}\) and \((f, r_2, u_0) = \Phi((u, \nabla p)) \in \mathcal{Y}\). Hence \(\mathcal{X} = \mathcal{Y}\).

Since \((f, 0, u_0) \in \mathcal{X} = \mathcal{Y}\) there exists \(p \in \mathcal{X}\) such that \(\Phi(p) = (f, 0, u_0)\), which defines a smooth solution of (7). Further since \(\Phi : \mathcal{X} \to \mathcal{Y}\) is a homeomorphism the solution is unique. The assertion of Theorem 2 follows.

We note that \(\{X_N\}_N\) such that \(\bigcup_{N} X_N = \mathcal{X}\) is not unique. In our case \(\{X_N\}_N\) is taken to be \(\{X^w_N\}_{p \in \mathcal{X}}\) and the existence and uniqueness of local solutions are obtained. In summary we used the following Navier-Stokes conditions (which are not assumptions) and conclude \(\Phi\) is a homeomorphism.

1. \(\Phi : \mathcal{X} \to \mathcal{Y}\) is a \(C^\infty\)-map such that \(d\Phi(p) : T_p \mathcal{X} \to T_{\Phi(p)} \mathcal{Y}\) is a linear isomorphism.

2. Each seminorm \(p_k\) of \(\mathcal{Y}\), which is induced from the relative topology from

\[
W^{k,k}_3 \times (W^{k,k+2}_3)_{\text{div}} \times H^{k+2}(\mathbb{R}^3/\mathbb{Z}^3)^3,
\] (34)

is actually a norm, that is, it satisfies the condition that \(p_k(r) = 0\) implies \(r = 0\).

3. The above holds if we replace \(I\) with \(J\).

4. For any \(r \in \mathcal{X}\) there exists \(q \in \bigcup_{p} U^J_p\) such that \(\Phi'(q)|_t = r(t)\) \((t_0 \leq t < t_0 + t_1)\) for small \(t_1\).
5. Let \((f, r_2, u_0) \in \mathcal{Z}\). Let \(x \in \mathbb{R}^3 / \mathbb{Z}^3\). Then there exist a compact neighbourhood \(K_x\) of \(x\) and \((u^{K_x}, \nabla p^{K_x}) \in \mathcal{K}_x\) satisfying equation (7) on \(I \times K_x\).

From Theorem 2 it is easy to prove the following corollary.

**Corollary 3.** Let \(\nu > 0\). Let \(I := [0, \infty)\). Let \(u_0 \in C^\infty(\mathbb{R}^3 / \mathbb{Z}^3)^3\) such that \(\nabla \cdot u_0 = 0\). Then there exists an unique \((u, \nabla p) \in W_3 \times (W_1)_\nu\) such that

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= -(u \cdot \nabla)u + \nu \Delta u - \nabla p, \\
\nabla \cdot u &= 0, \\
|u|_{t=0} &= u_0.
\end{aligned}
\] (35)

In above theorems the smooth solution may be expanded into an appropriate series, which enables us to numerically simulate any flow, especially any turbulence. How to deal with a turbulence or the chaos may be determined so that any possible state within the indeterminacy (e.g. within an error of measurement) is dealt, which is necessary and sufficient if the indeterminacy may not be removed. Actually however this may form an infinite set of algorithms, so we develop a theory of artificial intelligence to carry out these algorithms (which are in the direction of our rubric) automatically. It is worth mentioning that our artificial intelligence has a generality and the generality makes it possible to solve the problem.

A deterministic Turing Machine is usually used for the proof in science and technology. Thus a long and complicated solution is widely celebrated. We however use a nondeterministic Turing Machine, of which features are as follows: (i) It is astronomically easy to read. (ii) It uses oracles without the details. The difficulty consists in the discovery so our paper is written in a quite different approach.

## 2 Preliminaries

**Definition 4.** Let \(Z\) be a Banach space. Let \(T_s\) \((s \geq 0)\) be a bounded linear operator on \(Z\). A C0-semigroup is a family \(\{T_s\}\) such that

\[
\begin{aligned}
T_{s_1} T_{s_2} &= T_{s_1 + s_2} \quad (s_1, s_2 \geq 0), \\
T_0 &= Id.
\end{aligned}
\] (36) (37)

and such that

\[
\lim_{s \to s_0} T_s x = T_{s_0} x
\] (38)

for each \(s_0 \geq 0\) and each \(x \in Z\).
Definition 5. Let $Z$ be a Banach space. Let $\{T_s\}$ be a $C^0$-semigroup on $Z$. The infinitesimal generator $A$ of $\{T_s\}$ is a linear operator of which domain is the set $D(A) := \{z \in Z \mid \lim_{s \to +0} \frac{1}{s}(T_s - I)z \text{ exists in } Z\}$ such that for $z \in D(A)$

$$Az := \lim_{s \to +0} \frac{1}{s}(T_s - I)z.$$  

(39)

Although it is not used we refer to the following Hille-Yosida theorem [8], Section 12.3, Theorem 12.3.2, of which expression is slightly changed.

Theorem 6. Let $Z$ be a Banach space. Let $A$ be a closed linear operator on $Z$. Then $A$ generates a $C^0$-semigroup if and only if the domain of $A$ is dense and there exist $M > 0$ and $\beta \in \mathbb{R}$ such that

$$|| (\lambda - A)^{-k} || \leq M(\lambda - \beta)^{-k} (\lambda > \beta), \quad (40)$$

for $k = 1, 2, \ldots$.

The following definition, of which expression is slightly changed, is given in [10].

Definition 7 (see [10]). Let $Z$ be a Banach space. Let $T > 0$. Assume a linear operator $A(t)$ on $Z$ generates a $C^0$-semigroup for each $t \in [0, T]$. The family $\{A(t)\}$ is stable if there exist $M > 0$ and $\beta \in \mathbb{R}$ such that

$$|| \prod_{j=1}^{k} (\lambda - A(t_j))^{-1} || \leq M(\lambda - \beta)^{-k} (\lambda > \beta), \quad (41)$$

for any finite family $\{t_j\}$ with $0 \leq t_1 \leq \cdots \leq t_k \leq T$ ($k = 1, 2, \ldots$). The product $\prod$ is time-ordered, i.e. a factor with larger $t_j$ stands to the left of ones with smaller $t_j$.

The following, of which expression is slightly changed, is given in [10], Section 3, Proposition 3.3, formula (3.2).

Lemma 8. Let $Z$ be a Banach space. Let $A(t)$ be a linear operator on $Z$ which generates a $C^0$-semigroup for each $t \in [0, T]$. If $\{A(t)\}$ is stable with $M > 0$ and $\beta \in \mathbb{R}$ then

$$|| \prod_{j=1}^{k} e^{s_j A(t_j)} || \leq M e^{\beta \left( \sum_{j=1}^{k} s_j \right)}, \quad (42)$$

for $0 \leq t_1 \leq \cdots \leq t_k \leq T$ and $s_j \geq 0$. The product $\prod$ is time-ordered, i.e. a factor with larger $t_j$ stands to the left of ones with smaller $t_j$.

The following is a special case of a result of [10].
Theorem 9 (T. Kato). Let $T > 0$. Let $Z$ be a Banach space. Let $A(t)$ be a linear operator on $Z$. Let the domain of $A(t)$ be equal to $Z$. Assume $A(t)$ generates a $C^0$-semigroup for each $t \in [0, T]$ and that $\{A(t)\}$ is stable. Let $f \in C^0([0, T], Z)$ and $u_0 \in Z$ then the equation

$$\begin{align*}
\frac{du}{dt} &= A(t)u + f, \\
u|_{t_0} &= u_0,
\end{align*}$$

has a solution $u \in C^1([0, T], Z)$.

Let $B(Z)$ be the set of bounded operators on $Z$.

Corollary 10. Let $Z$ be a Banach space. Let $t_0 < T_0$ (resp. $T_0 < t_0$). Assume $A(t) \in B(Z)$ for each $t \in [t_0, T_0]$ (resp. for each $t \in [T_0, t_0]$). Let $f \in C^0([t_0, T_0], Z)$ (resp. $f \in C^0([T_0, t_0], Z)$) and $u_0 \in Z$. Assume there exists $C > 0$ such that $\|A(t)|B(Z)\| \leq C$ for any $t \in [t_0, T_0]$ (resp. for any $t \in [T_0, t_0]$). Then the equation

$$\begin{align*}
\frac{du}{dt} &= A(t)u + f, \\
u|_{t_0} &= u_0,
\end{align*}$$

has a solution $u \in C^1([t_0, T_0], Z)$ (resp. $u \in C^1([T_0, t_0], Z)$).

The following is a special case of a result of [2].

Theorem 11 (K. Deimling). Let $Z$ be a Banach space. Let $t_0 < T_0$ (resp. $T_0 < t_0$). Assume $A(t) \in B(Z)$ for each $t \in [t_0, T_0]$ (resp. for each $t \in [T_0, t_0]$). Let $f \in C^0([t_0, T_0], Z)$ (resp. $f \in C^0([T_0, t_0], Z)$) and $u_0 \in Z$. Assume there exists $C > 0$ such that $\|A(t)|B(Z)\| \leq C$ for any $t \in [t_0, T_0]$ (resp. for any $t \in [T_0, t_0]$). Then the equation

$$\begin{align*}
\frac{du}{dt} &= A(t)u + f, \\
u|_{t_0} &= u_0,
\end{align*}$$

has at most one solution $u \in C^1([t_0, T_0], Z)$ (resp. $u \in C^1([T_0, t_0], Z)$).

The following theorem, of which expression is slightly changed, is given in [11], Chapter I, section 5, Theorem 5.2.

Theorem 12. Let $X, Y$ be Banach spaces, $O$ an open subset of $X$ and let $\xi : O \rightarrow Y$ a $C^p$-morphism with $p \in \mathbb{N}$. Assume that for a point $a_0 \in O$ the derivative $d\xi(a_0) : X \rightarrow Y$ is a topological linear isomorphism. Then $\xi$ is a local $C^p$-isomorphism at $a_0$.

3 Nonlinear ordinary differential equations (alternate proof)

We give an alternate proof of the following well-known theorem (see e.g. [3]).
Theorem 13. Let $X$ be a Banach space. Let $f : \mathbb{R} \times X \ni (t, u_1) \mapsto f(t, u_1) \in X$ be $C^0$ in $t$ and $C^1$ in $u_1$. Let $u \in X$ be arbitrary. Let the Fréchet derivative $D_x f(t, u)$ of $f(t, \cdot)$ at $u$ be $C^0$ in $t$. Then the equation

\[
\begin{cases}
\frac{du}{dt} = f(t, u), \\
u|_{t=0} = u_0,
\end{cases}
\]

has an unique solution $u \in C^1(I_1, X)$ for some $I_1 = (a, b) (a < 0 < b)$ that is maximal in this property.

Let $J$ a compact interval in $\mathbb{R}$ containing $t_0 \in \mathbb{R}$. Let $u \in C^1(J, X)$. We prove the following lemma.

Lemma 14. The map

\[
h \mapsto \left[ \frac{dh}{dt} - D_x f(t, u)h \right]_{h|_{t=t_0}}
\]

is a topological linear isomorphism from $C^1(J, X)$ to $C^0(J, X) \times X$.

Proof. Consider the equation

\[
\begin{cases}
\frac{dh}{dt} = D_x f(t, u)h + r, \\
h|_{t=t_0} = h_0,
\end{cases}
\]

where $r \in C^0(J, X)$, $h_0 \in X$. Let $B(X)$ be the set of bounded operators on $X$. Then by definition $D_x f(t, u) \in B(X)$ for each $t \in J$ and there exists $C > 0$ such that $\|D_x f(t, u)\|_{B(X)} \leq C$ for any $t \in J$. Thus by Corollary 10 and Theorem 11 equation (48) has an unique solution in $C^1(J, X)$. The map

\[
h \mapsto \left[ \frac{dh}{dt} - D_x f(t, u)h \right]_{h|_{t=t_0}}
\]

is a continuous linear isomorphism from $C^1(J, X)$ to $C^0(J, X) \times X$ and by the open mapping principle a topological linear isomorphism. The assertion follows. □

Let $\mathcal{X}^J := C^1(J, X)$, $\mathcal{Y}^J := C^0(J, X) \times X$. Let $p := u \in \mathcal{X}^J$. Define

\[
\Phi^J : \mathcal{X}^J \ni u_1 \mapsto (F^J(u_1), G^J(u_1)) \in \mathcal{Y}^J,
\]

where

\[
\begin{cases}
F^J(u_1) := \frac{du_1}{dt} - f(t, u_1), \\
G^J(u_1) := u_1|_{t=t_0},
\end{cases}
\]

$F^J, G^J$ are $C^1$ in $u_1$. At $p = u \in \mathcal{X}^J$, by calculation

\[
d\Phi^J(p) : h \mapsto \left[ \frac{dh}{dt} - D_x f(t, u)h \right]_{h|_{t=t_0}}.
\]
By Lemma 14 $d\Phi^J(p)(h)$ ($p \in \mathcal{X}^J$) is a topological isomorphism from the
tangent space $T_p \mathcal{X}^J$ of $\mathcal{X}^J$ at $p$ to the tangent space $T_{\Phi^J(p)} \mathcal{Y}^J$ of $\mathcal{Y}^J$ at
$\Phi^J(p)$. Thus by Theorem 12 $\Phi^J$ is a local diffeomorphism $(X = O = \mathcal{X}^J, Y = \mathcal{Y}^J, \xi = \Phi^J, a_0 = p)$.

We prove the local existence and uniqueness of a solution of equation (46). Let $U_{t_0}, \mathcal{C}_{t_0}$ be the inductive limits of $\mathcal{X}^J, \mathcal{Y}^J$ for $J \supseteq t_0$. Introduce to
them the natural topologies (the quotient topologies induced from $\prod J \to U_{t_0}$
and $\prod J \to \mathcal{C}_{t_0}$) so that $\{\Phi^J\}_J$ induces a local diffeomorphism $\Phi_{t_0}, U_{t_0}$ is
connected and $\mathcal{C}_{t_0}$ is simply connected.

**Lemma 15.** $\Phi_{t_0}$ is surjective (and thus $\Phi_{t_0}$ is a homeomorphism).

**Proof.** Let $(r, u_0) \in \mathcal{Y}^J$. From the definition
$$
\mathcal{X}^J \ni u_1 \mapsto (F^J(u_1)|_{t_0}, G^J(u_1)) \in X \times X
$$
(53)
is surjective. Thus there exists $u' \in \mathcal{Y}^J$ such that $(F^J(u')|_{t_0}, G^J(u')) = (r|_{t_0}, u_0)$. Since $\Phi^J$ is an open map there exists $\epsilon > 0$ such that any $y \in \mathcal{Y}^J$
with
$$
||y - (F^J(u'), G^J(u'))||_{C^0(J,X \times X} < \epsilon
$$
is in $\text{Im}\Phi^J$. On the other hand for $s \in \mathbb{R}$ with $|s|$ small
$$
||(r|_{t=t_0+s}, u_0) - (F^J(u')|_{t=t_0+s}, G^J(u'))||_{X \times X} < \epsilon.
$$
(55)
Thus there exists $u \in \mathcal{X}^J$ such that $(F^J(u)|_t, G^J(u)) = (r|_t, u_0)$ on a neighbour-
bourhood of $t_0$. The image of $\Phi_{t_0}$ contains the equivalence class of $(r, u_0)$ at $t_0$.
Hence $\Phi_{t_0}$ is surjective. The assertion follows.

**Remark 16.** The above results are also proved by successive approximation
method.

**Proof of Theorem 13.** Now we obtain a bijection $\tilde{\Phi} : \mathcal{U} = \prod_{t_0} U_{t_0} \to \mathcal{C} = \prod_{t_0} \mathcal{C}_{t_0}$ and introducing a sheaf structure to $\mathcal{C}$ (induced from $\mathcal{U}$) through this
bijection a sheaf isomorphism. Thus for $u_0 \in X$ there exists a section $\tilde{p}$ defined
on a maximal $I_1 := (a, b)$ such that
$$
\tilde{\Phi}(\tilde{p}) = (\frac{d\tilde{p}}{dt} - f(t, \tilde{p}), \tilde{p}|_{t=0}) = (0, u_0),
$$
(56)
where $\tilde{p}$ is locally unique because $\tilde{\Phi}$ is an isomorphism. Hence $\tilde{p}$ is an unique so-
lution of equation (46) defined on an interval $I_1$ that is maximal in this property.
The assertion follows. \qed
4 Hodge Theory on $\mathbb{R}^3/\mathbb{Z}^3$

Let $\nabla \cdot (\cdot) : u \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3 \mapsto \nabla \cdot u \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3)$. Let

\[
(C^\infty(\mathbb{R}^3/\mathbb{Z}^3))^\nabla := \{v \mid v = \nabla w \ (\exists w \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3))\}.
\] (57)

The following is a special case of [4], Chapter VI, Section 3.3, (3.16) Theorem.

**Theorem 17.** $C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3 = \text{Ker}(\nabla \cdot (\cdot)) \oplus (C^\infty(\mathbb{R}^3/\mathbb{Z}^3))^\nabla$.

5 Linear evolution equations on $\mathbb{R}^3/\mathbb{Z}^3$

The following theorem, of which expression is slightly changed, is also known as Hille-Yosida theorem (see [20], Chapter IX, Section 8).

**Theorem 18.** Let $X$ be a Banach space and $\tilde{A}$ a linear operator on $X$. Assume the domain of $\tilde{A}$ is dense in $X$. Then $\tilde{A}$ generates a contraction $C^0$-semigroup if and only if $\tilde{A}$ is dissipative with respect to a semiscalar product and the range of $\text{Id} - \tilde{A}$ is equal to $X$.

Let $\mathcal{P} : C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3 \to \text{Ker}(\nabla \cdot (\cdot))$ be the projection. For $n \in \mathbb{N}$ let

\[
W_n := C^\infty(I, C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^n).
\] (58)

We shall prove the existence and uniqueness of a solution of the equation

\[
\begin{aligned}
\dot{h} + \mathcal{P}((u \cdot \nabla)\mathcal{P}h + (\mathcal{P}h \cdot \nabla)u - \nu \Delta \mathcal{P}h) - g &= 0, \\
h(0) &\in C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3,
\end{aligned}
\] (59)

for $g \in W_3$.

Let $k \in \mathbb{Z}_{\geq 0}$. Let $I := [0, T]$ and $u \in W_3$. Let $H^k(\mathbb{R}^3/\mathbb{Z}^3)^3$ be the Sobolev space. Define a linear operator $A'_k(t)$ on $H^k(\mathbb{R}^3/\mathbb{Z}^3)^3$ by

\[

A'_k(t)h := \mathcal{P}((u \cdot \nabla)\mathcal{P}h + (\mathcal{P}h \cdot \nabla)u - \nu \Delta \mathcal{P}h)
\] (60)

for $h \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3$. Since the adjoint $(A'_k(t))^*$ of $A'_k(t)$ is densely defined $A'_k(t)$ is closable. Let $A_k(t)$ be the closure of $A'_k(t)$. For $k_1, k_2 \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ let

\[
W_n^{k_1, k_2} := C^{k_1}(I, H^{k_2}(\mathbb{R}^3/\mathbb{Z}^3)^n).
\] (61)

**Lemma 19.** Let $I := [0, T]$ $(T > 0)$. Let $\nu > 0$. Assume $u \in W_3$. Then the equation

\[
\dot{h} - A_k(t)h = 0
\] (62)

has at most one solution $h \in W_n^{1, k}$ for any initial condition $h(0) \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3$. 

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Proof. Let \( h \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3 \). Observe that 
\[-A_k(t)h = \mathcal{P}(-\nu(\nabla - A(t))^2)\mathcal{P}h + \mathcal{P}\mathbb{E}(t)\mathcal{P}h\]
for some \( W_1 \)-coefficiential \( 3 \times 1 \) and \( 3 \times 3 \) matrices \( A(t), \mathbb{E}(t) \). Thus by an elementary argument

\[
\text{Re} < -A_k(t)h, h >_{H^k(\mathbb{R}^3/\mathbb{Z}^3)^3} \geq -c < h, h >_{H^k(\mathbb{R}^3/\mathbb{Z}^3)^3}
\]  

for some \( c > 0 \). Taking limit formula (63) also holds for \( h \in D(-A_k(t)) \), where 
\( D(-A_k(t)) \) is the domain of \( -A_k(t) \).

Let \( h \) be the solution of equation (62) and \( H = e^{-ct}h \). Observe that \( h \) depends on \( t \). Then

\[
< \dot{H}, H >_{H^k(\mathbb{R}^3/\mathbb{Z}^3)^3} = (A_k(t) - c)H, H >_{H^k(\mathbb{R}^3/\mathbb{Z}^3)^3}.
\]  

Assume \( h(0) = 0 \). Let \( t_0 \in [0, T] \). By formula (63)

\[
||H(t_0)||^2_{H^k(\mathbb{R}^3/\mathbb{Z}^3)^3} - ||H(0)||^2_{H^k(\mathbb{R}^3/\mathbb{Z}^3)^3} = 2 \int_0^{t_0} \text{Re} < (A_k(t) - c)H, H >_{H^k(\mathbb{R}^3/\mathbb{Z}^3)^3} \ dt \leq 0.
\]  

It follows that

\[
||H(t_0)||^2_{H^k(\mathbb{R}^3/\mathbb{Z}^3)^3} \leq ||H(0)||^2_{H^k(\mathbb{R}^3/\mathbb{Z}^3)^3} = 0.
\]  

Hence \( h(t_0) = 0 \). Since \( t_0 \in [0, T] \) is arbitrary \( h = 0 \). Assume \( h_1, h_2 \) are two solutions. Then \( h_2(0) - h_1(0) = 0 \) and \( h_2 - h_1 \) is a solution of the equation (62). Thus the above argument shows that \( h_2 - h_1 = 0 \). From this the assertion follows.

Lemma 20. \( A_k(t) \) generates a \( C^0 \)-semigroup for each \( t \in I \).

Proof. Take \( c \) as in Lemma 19. Since by definition the domain of \( A_k(t) \) is dense, so is that of \( (A_k(t) - c) \). Observe that

\[
\text{Re} < (A_k(t) - c)h, h >_{H^k(\mathbb{R}^3/\mathbb{Z}^3)^3} \leq 0
\]  

for all \( h \in D(A_k(t) - c) \), where \( D(A_k(t) - c) \) is the domain of \( (A_k(t) - c) \), so \( (A_k(t) - c) \) is dissipative. In particular the image of \( \text{Id} - (A_k(t) - c) \) is closed. The adjoint operator \( (\text{Id} - (A_k(t) - c))^* \) of \( (\text{Id} - (A_k(t) - c) \) is clearly injective. Hence the image of \( \text{Id} - (A_k(t) - c) \) is the whole space \( H^k(\mathbb{R}^3/\mathbb{Z}^3)^3 \). By Theorem 18, \( (A_k(t) - c) \) generates a contraction \( C^0 \)-semigroup. The assertion follows.

Lemma 21. Let \( k, k' \in \mathbb{Z}_{>0} \). Let \( \{e^{sA_k(t)}\}_{s \geq 0} \) be a \( C^0 \)-semigroup on \( H^k(\mathbb{R}^3/\mathbb{Z}^3)^3 \) generated by \( A_k(t) \) for each \( t \). Then \( e^{sA_k(t)}h_0 = e^{sA_{k'}(t)}h_0 \) for any \( h_0 \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3 \) and \( s \in I := [0, T] \).
Proof. Assume without loss of generality \( k \leq k' \). Observe that by assumption \( h_0 \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3 \) so that \( h_1(s, x) := e^{sA_k(t)}h_0 \in W^{1,k}_3 \) and \( h_2(s, x) := e^{sA_{k'}(t)}h_0 \in W^{1,k'}_3 \). Then since \(-A_k(t)|_{H^k(\mathbb{R}^3/\mathbb{Z}^3)^3} = -A_{k'}(t)\) it follows that \( h_1 \) and \( h_2 \) are solutions of

\[
\begin{cases}
\frac{dh}{dt} + A_k(t)h = 0, \\
h(0) = h_0.
\end{cases}
\]  

(70)

Since \( A_k(t) \) is dissipative an elementary argument shows that the solution \( h \in W^{1,k}_3 \) of this equation is unique. Thus \( e^{sA_k(t)}h_0 = h_1(s, x) = h_2(s, x) = e^{sA_{k'}(t)}h_0 \) for \( s \in I \). The assertion follows. □

Let \( B(H^k(\mathbb{R}^3/\mathbb{Z}^3)^3) \) be the set of continuous linear operators on \( H^k(\mathbb{R}^3/\mathbb{Z}^3)^3 \).

**Theorem 22.** Assume

\[
\| \prod_{j=1}^J (\lambda - A_k(t_j))^{-1} \|_{B(H^k(\mathbb{R}^3/\mathbb{Z}^3)^3)} \leq M_k(\lambda - \beta_k)^j (\lambda > \beta_k).
\]  

(71)

for \( 0 \leq t_1 \leq \cdots \leq t_J \leq T \) \((J = 1, 2, \ldots)\), where the product \( \prod \) is time-ordered, i.e. a factor with larger \( t_j \) stands to the left of ones with smaller \( t_j \). Then there exists an operator-valued function \( U(t, s) \) \((0 \leq s \leq t \leq T)\) on \( C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3 \) that satisfies the following:

(a) \((s, t) \mapsto U(t, s)h_0 \) \((h_0 \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3)\) is continuous, \( U(s, s) = \text{Id} \),

\[
\|U(t, s)\|_{B(H^k(\mathbb{R}^3/\mathbb{Z}^3)^3)} \leq M_k e^{\beta_k(t-s)}.
\]  

(72)

(b) If \( s \leq r \leq t \) then \( U(t, s) = U(t, r)U(r, s) \).

(c) For \( h_0 \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3 \) and \( s \in [0, T) \),

\[
D^+ U(t, s)h_0|_{t=s} = A_0(s)h_0,
\]  

(73)

where \( D^+ \) denotes the right derivative.

(d) For \( h_0 \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3 \) and \( 0 \leq s \leq t \leq T \),

\[
\frac{\partial}{\partial s} U(t, s)h_0 = -U(t, s)A_0(s)h_0.
\]  

(74)

(e) For \( h_0 \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3 \) and \( 0 \leq s \leq t \leq T \),

\[
\frac{\partial}{\partial t} U(t, s)h_0 = A_0(t)U(t, s)h_0.
\]  

(75)

**Proof.** By Lemma 20, \( A_k(t) \) generates a \( C^0\)-semigroup \( \{e^{sA_k(t)}\}_{s \geq 0} \) and since \( C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3 \) is dense in \( H^k(\mathbb{R}^3/\mathbb{Z}^3)^3 \), it is obtained by Lemma 21 that \( e^{sA_0(t)}|_{H^k(\mathbb{R}^3/\mathbb{Z}^3)^3} = e^{sA_k(t)} \).

\[
e^{sA_0(t)}|_{H^k(\mathbb{R}^3/\mathbb{Z}^3)^3} = e^{sA_k(t)}.
\]  

(76)
By Lemma 8,

\[ \| \prod_{j=1}^{J} e^{s_j \lambda(t_j)} \|_{H^k(\mathbb{R}^3/\mathbb{Z}^3)}^3 \leq \beta_k \left( \sum_{j=1}^{J} s_j \right) \leq M_k e^{\beta_k(t-s)} \]  

(79)

for \( 0 \leq t_1 \leq \cdots \leq t_J \leq T \) and \( s_j \geq 0 \). The product \( \prod \) is time-ordered, i.e. a factor with larger \( t_j \) stands to the left of ones with smaller \( t_j \).

Observe that \( A_0(t)|_{H^k(\mathbb{R}^3/\mathbb{Z}^3)} = A_k(t) \). Let \( L(H^k+2(\mathbb{R}^3/\mathbb{Z})^3, H^k(\mathbb{R}^3/\mathbb{Z}^3)) \) be the set of continuous linear operators from \( H^k+2(\mathbb{R}^3/\mathbb{Z}^3) \) to \( H^k(\mathbb{R}^3/\mathbb{Z}^3) \). Let \( A_{0,n}(t) = A_0(T|nt/T|/n), \) where \([\cdot]\) denotes the Gauss symbol. Then

\[ \|A_{0,n}(t) - A_0(t)\|_{L(H^k+2(\mathbb{R}^3/\mathbb{Z}^3), H^k(\mathbb{R}^3/\mathbb{Z}^3))} \rightarrow 0 \ (n \rightarrow \infty). \]  

(80)

Let

\[ U_n(t) = e^{i(t-s)A_0(k'T/n)}, \ (k'T/n \leq s \leq (k'+1)T/n), \]  

(81)

\[ U_n(t, s) = e^{i(t-l'T/n)A_0(l'T/n)}e^{i(t'-l'T/n)A_0((l'-1)T/n)} \cdots e^{i(T/n)A_0((k'+1)T/n)}e^{i((k'+1)T/n-s)A_0(k'T/n)}, \]  

(82)

\[ (k'T/n \leq s < (k'+1)T/n, l'T/n \leq t < (l'+1)T/n, k' < l'). \]  

(83)

By inequality (77)-(79),

\[ \|U_n(t, s)\|_{B(H^k(\mathbb{R}^3/\mathbb{Z}^3))} \leq M_k e^{\beta_k(t-s)}. \]  

(85)

Thus \( U_n(t, s) \) satisfies (a), (b). For \( h_0 \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3) \) and \( t \neq k''T/n \) \( (k'' = 0, 1, \ldots, n) \),

\[ \frac{\partial}{\partial t} U_n(t, s) h_0 = A_{0,n}(t) U_n(t, s) h_0, \]  

(86)

and for \( h_0 \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3) \) and \( s \neq l''T/n \) \( (l'' = 0, 1, \ldots, n) \),

\[ \frac{\partial}{\partial s} U_n(t, s) h_0 = -U_n(t, s) A_{0,n}(s) h_0. \]  

(87)
Let $h_0 \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3$. Then by inequality (85),

\[
||U_n(t,s)h_0 - U_m(t,s)h_0||_{H^k(\mathbb{R}/\mathbb{Z}^3)^3} = || - \int_s^t \frac{\partial}{\partial r} U_n(t,r) U_m(r,s) h_0 dr||_{H^k(\mathbb{R}/\mathbb{Z}^3)^3},
\]

(88)

\[
= || \int_s^t U_n(t,r) (A_{0,n}(r) - A_{0,m}(r)) U_m(r,s) h_0 dr||_{H^k(\mathbb{R}/\mathbb{Z}^3)^3}
\]

(89)

\[
\leq M_k M_{k+2} e^{\max\{\beta_k, \beta_{k+2}\}(t-s)} ||h_0||_{H^{k+2}(\mathbb{R}/\mathbb{Z}^3)^3}
\]

(90)

\[
\times \int_s^t ||A_{0,n}(r) - A_{0,m}(r)||_{L(H^{k+2}(\mathbb{R}/\mathbb{Z}^3)^3, H^k(\mathbb{R}/\mathbb{Z}^3)^3)} dr
\]

(91)

\[
\rightarrow 0 \quad (n, m \rightarrow \infty).
\]

(92)

k \in \mathbb{Z}_{\geq 0} is arbitrary and thus for any $h_0 \in C^\infty(\mathbb{R}/\mathbb{Z}^3)^3$ and $0 \leq s \leq t \leq T$,

\[
U(t,s)h_0 := \lim_{n \rightarrow \infty} U_n(t,s)h_0
\]

(94)

exists. Note that $(s,t) \mapsto U(t,s)$ is continuous because the convergence is uniform in $0 \leq s \leq t \leq T$. Since $U_n(t,s)$ satisfies (a), (b) so does $U(t,s)$. For $h_0 \in C^\infty(\mathbb{R}/\mathbb{Z}^3)^3$,

\[
||U_n(t,s)h_0 - e^{(t-s)A_0(s)}h_0||_{H^k(\mathbb{R}/\mathbb{Z}^3)^3} = || - \int_s^t \frac{\partial}{\partial r} U_n(t,r) e^{(r-s)A_0(s)} h_0 dr||_{H^k(\mathbb{R}/\mathbb{Z}^3)^3},
\]

(95)

\[
= || \int_s^t U_n(t,r) (A_{0,n}(r) - A_0(r)) e^{(r-s)A_0(s)} h_0 dr||_{H^k(\mathbb{R}/\mathbb{Z}^3)^3}
\]

(96)

\[
\leq M_k M_{k+2} e^{\max\{\beta_k, \beta_{k+2}\}(t-s)} ||h_0||_{H^{k+2}(\mathbb{R}/\mathbb{Z}^3)^3}
\]

(97)

\[
\times \int_s^t ||A_{0,n}(r) - A_0(r)||_{L(H^{k+2}(\mathbb{R}/\mathbb{Z}^3)^3, H^k(\mathbb{R}/\mathbb{Z}^3)^3)} dr.
\]

(98)

It follows that for $h_0 \in C^\infty(\mathbb{R}/\mathbb{Z}^3)^3$,

\[
||U(t,s)h_0 - e^{(t-s)A_0(s)}h_0||_{H^k(\mathbb{R}/\mathbb{Z}^3)^3} \leq M_k M_{k+2} e^{\max\{\beta_k, \beta_{k+2}\}(t-s)} ||h_0||_{H^{k+2}(\mathbb{R}/\mathbb{Z}^3)^3}
\]

(99)

\[
\times \int_s^t ||A_{0,n}(r) - A_0(r)||_{L(H^{k+2}(\mathbb{R}/\mathbb{Z}^3)^3, H^k(\mathbb{R}/\mathbb{Z}^3)^3)} dr.
\]

(100)

Since $k \in \mathbb{Z}_{\geq 0}$ is arbitrary this proves (c). Similarly it is shown that for $h_0 \in C^\infty(\mathbb{R}/\mathbb{Z}^3)^3$, $s < t$ it is obtained

\[
D^- U(t,s)h_0|_{s=t} = -A_0(t)h_0.
\]

(101)

where $D^-$ denotes the left derivative. For $h_0 \in C^\infty(\mathbb{R}/\mathbb{Z}^3)^3$, $s < t$ it is obtained
by (c) and the continuity of $U(t,s)$ that

$$
\frac{1}{\epsilon}(U(t, s + \epsilon)h_0 - U(t, s)h_0)
\quad (104)
$$

$$
= U(t, s + \epsilon)\frac{1}{\epsilon}(h_0 - U(s + \epsilon, s)h_0)
\quad (105)
$$

$$
\rightarrow -U(t, s)A_0(s)h_0 \ (\epsilon \to +0).
\quad (106)
$$

For $h_0 \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3$, $s \leq t$ it is obtained by formula (103) that

$$
\frac{1}{\epsilon}(U(t, s)h_0 - U(t, s - \epsilon)h_0)
\quad (107)
$$

$$
= U(t, s)\frac{1}{\epsilon}(h_0 - U(s, s - \epsilon)h_0)
\quad (108)
$$

$$
\rightarrow -U(t, s)A_0(s)h_0 \ (\epsilon \to +0).
\quad (109)
$$

They prove (d). (e) follows from (c) and (d).

**Lemma 23.** Let $I := [0, T]$ $(T > 0)$. Let $\nu > 0$. Assume $u, g \in W_3$. Then the equation

$$
\dot{h} - A_0(t)h - g = 0
\quad (110)
$$

has a solution $h \in W_3^{1,\infty}$ that is uniquely determined from $h(0) \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3$.

**Proof.** By Theorem 22 there exists $U(t,s)$ satisfying (a)-(e). Let

$$
h := U(t,0)h(0) + \int_0^t U(t,s)g(s)ds.
\quad (111)
$$

Then $h \in W_3^{1,\infty}$ and it is a solution of equation (110). Let $h_1, h_2$ be two solutions of equation (110) then $h_2 - h_1$ is a solution of

$$
\begin{cases}
\dot{h} - A_0(t)h = 0, \\
h|_{t=0} = 0.
\end{cases}
\quad (112)
$$

By Lemma 19 this equation has the unique solution 0. Thus $h_2 - h_1 = 0$ and equation (110) has a solution $h \in W_3^{1,\infty}$ that is uniquely determined from $h(0) \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3$. The assertion follows.

**Theorem 24.** Let $I := [0, T]$ $(T > 0)$. Let $\nu > 0$. Assume $u, g \in W_3$. Then the equation

$$
\dot{h} + \mathcal{P}((u \cdot \nabla)\mathcal{P}h + (\mathcal{P}h \cdot \nabla)u - \nu \Delta \mathcal{P}h) - g = 0
\quad (113)
$$

has a solution $h \in W_3$ that is uniquely determined from $h(0) \in C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3$.

**Proof.** By Lemma 23, equation (113) has an unique solution $h$ in $W_3^{1,\infty}$. Since $h$ satisfies equation (113), if $h \in W_3^{l,\infty}$ for $l \in \mathbb{N}$, it follows that $\dot{h} \in W_3^{l,\infty}$, that is, $h \in W_3^{l+1,\infty}$. Hence by induction it is proved that $h \in W_3$. The assertion follows.
6 Open maps between Fréchet spaces

Definition 25. Let $E$ be a Hausdorff topological linear space. Assume the topology is defined by countably many seminorms on $E$. Let $d$ be a translation invariant complete metric on $E$ induced by the seminorms. The pair $(E, d)$ is called a Fréchet space. We also denote the Fréchet space by $E$.

Definition 26. Let $E$ be a Fréchet space. A map $\psi : E \to E$ is a contraction map if there exist $0 < K < 1$ such that

$$d(\psi(e_1), \psi(e_2)) \leq K \cdot d(e_1, e_2)$$

(114)

for any $e_1, e_2 \in E$.

Lemma 27. Let $E$ be a Fréchet space. Then a contraction map $\psi : E \to E$ has an unique fixed point.

Proof. We first show the existence. Let $e \in E$. Consider $\{\psi^n(e)\}_N$. By inequality (114) this is a Cauchy sequence with respect to $d$. Thus since $E$ is complete it is convergent. It is easy to show that the limit is a fixed point. The uniqueness follows from inequality (114). The assertion follows. $\square$

Lemma 28. Let $E$ be a Fréchet space and let $\Psi(e) := e + \psi(e)$ ($e \in E$), where $\psi$ is a contraction map. Let $r > 0$ and $0 < s < 1$. Assume that $\psi(0) = 0$ and that

$$d(\psi(e_1), \psi(e_2)) < sr$$

(115)

for all $e_1, e_2 \in \{e \in E \mid d(0, e) < r\}$. If $e' \in E$ and $d(0, e') < (1 - s)r$ then there exists an unique $e \in \{e \in E \mid d(0, e) < r\}$ such that $\Psi(e) = e'$.

Proof. Let $e' \in E$ and $d(0, e') < (1 - s)r$. Let $g_{e'}(e) := e' - \psi(e)$ ($e \in E$). Then by assumption $g_{e'}$ is a contraction map on $\{e \in E \mid d(0, e) < r\}$ and by Lemma 27 there exists an unique fixed point $e \in \{e \in E \mid d(0, e) < r\}$. The assertion follows from this. $\square$

Definition 29. Let $\mathcal{X}, \mathcal{Z}$ be Fréchet spaces. Let $\Phi : \mathcal{X} \to \mathcal{Z}$ be a map. The Fréchet derivative $d\Phi(p)$ of $\Phi$ at $p \in \mathcal{X}$ is defined to be $A$ such that for any neighbourhoods $V$ and $W$ of $0 \in \mathcal{X}$ and $0 \in \mathcal{Z}$ and for any $q \in p + tV$ ($t > 0$)

$$\Phi(q) - \Phi(p) - A(q - p) \in o(t)W \ (t \to 0).$$

(116)

$\Phi$ is a $C^1$-map if the Fréchet derivative $d\Phi(p)$ exists for any $p \in \mathcal{X}$ and the map $p \mapsto d\Phi(p)$, which is the Fréchet derivative of $\Phi$, is continuous.

Lemma 30. Let $(\mathcal{X}, d_1)$, $(\mathcal{Z}, d_2)$ be Fréchet spaces. A $C^1$-map $\Phi : \mathcal{X} \to \mathcal{Z}$ such that $d\Phi(p)$ is a topological linear isomorphism for each $p \in \mathcal{X}$ is an open mapping.
Proof. By assumption $d\Phi(p)$ is a topological linear isomorphism from $T_p\mathcal{X}$ to $T_{\Phi(p)}\mathcal{Y}$ for each $p \in \mathcal{X}$. Let $p \in \mathcal{X}$ and identify neighbourhoods $\mathcal{X}_p$ and $\mathcal{Y}_{\Phi(p)}$ of $p \in \mathcal{X}$ and $\Phi(p) \in \mathcal{Y}$ with the corresponding neighbourhoods $\mathcal{X}_p$ and $\mathcal{Y}_{\Phi(p)}$ of $0 \in T_p\mathcal{X}$ and $0 \in T_{\Phi(p)}\mathcal{Y}$. Then

$$(d\Phi(p))^{-1} \circ \Phi(q) = q + \varphi(q) \quad (q \in \mathcal{X}_p),$$

where $\varphi$ is a contraction map of $\mathcal{X}_p$ and $d_1(0, \varphi(q)) = o(d_1(0, q))$. Let $r > 0$ be sufficiently small then by definition $\varphi(0) = 0$ and there exist $0 < s < 1$ such that

$$d_1(\varphi(e_1), \varphi(e_2)) < sr$$

for all $e_1, e_2 \in \{e \in E \mid d_1(0, e) < r\}$. By Lemma 28

$$(d\Phi(p))^{-1} \circ \Phi(\{e \in E \mid d_1(0, e) < r\}) \supset \{e \in E \mid d_1(0, e) < (1 - s)r\}.$$  

Here $d\Phi(p)$ is a topological isomorphism. Since $p \in \mathcal{X}$ is arbitrary it follows that $\Phi$ is an open mapping. The assertion follows.

For later use we give the following definition.

**Definition 31.** Let $\mathcal{X}, \mathcal{Y}$ be Fréchet spaces. Let $\Phi : \mathcal{X} \to \mathcal{Y}$ be a map. Let $k \in \mathbb{N}$. The Fréchet derivative $d^{(k)}\Phi$ of $\Phi$ of order $k$ is defined to be the Fréchet derivative of $d^{(k-1)}\Phi$. $\Phi$ is $C^\infty$ if the Fréchet derivative of $\Phi$ of any order exists and is continuous.

## 7 Navier-Stokes equations on $\mathbb{R}^3/\mathbb{Z}^3$

We prove Theorem 2. Let $\nu > 0$ and $I := [0, T]$ $(T > 0)$. Let

$$\Phi : W_3 \times (W_1_\nu) \to W_3 \times (W_3_\text{div}) \times C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3$$

be given by

$$(u, P) \mapsto \begin{bmatrix} u + (u \cdot \nabla)u - \nu \Delta u + P \\ \nabla \cdot u \\ u(0) \end{bmatrix},$$

where

$$(W_1_\nu) := \{v \mid v = \nabla w \ (\exists w \in W_1)\},$$

and

$$(W_3_\text{div}) := \{v \mid v = \nabla \cdot w \ (\exists w \in W_3)\}.$$  

We introduce to $(W_1_\nu)$ the relative topology as a subset of $W_3$ and to $(W_3_\text{div})$ the relative topology as a subset of $W_1$. Then $\Phi$ is $C^\infty$ in $(u, P)$ in the sense of Fréchet derivatives (see Definition 31). We begin with proving the following lemma:
Lemma 32. The map
\[ d\Phi((u, P)) : (h, \beta) \mapsto \begin{bmatrix} \dot{h} + (u \cdot \nabla) h + (h \cdot \nabla) u - \nu \Delta h + \beta \\ \nabla \cdot h \\ h(0) \end{bmatrix} \]  
(124)
is a linear isomorphism from
\[ W_3 \times (W_1)_\nabla \]  
(125)
to
\[ \{(a, b, c) \in W_3 \times (W_3)_{\text{div}} \times C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3 | \nabla \cdot c = b(0)\}. \]  
(126)

Proof. By calculation \(d\Phi\) is given by the map (124). Let
\[(a, b, c) \in \{(a, b, c) \in W_3 \times (W_3)_{\text{div}} \times C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3 | \nabla \cdot c = b(0)\}. \]  
(127)
Let
\[ \dot{h} + (u \cdot \nabla) h + (h \cdot \nabla) u - \nu \Delta h + \beta = a, \]  
(128)
\[ \nabla \cdot h = b, \]  
(129)
\[ h(0) = c. \]  
(130)

From equation (129), \( h \in W_3 \) is determined up to \( \text{Ker}(\nabla \cdot \cdot) \). Since \( \beta \in (W_1)_{\nabla} \), by Theorem 17, Theorem 24 and an elementary argument \( h \in W_3 \) is uniquely determined from equation (128) and equation (130). Then from equation (128), \( \beta \in (W_1)_{\nabla} \) is uniquely determined. From this the assertion follows. \( \square \)

Let
\[ \mathcal{X} := W_3 \times (W_1)_{\nabla} \]  
(131)
and
\[ \mathcal{Z} := \{r := (f, r_2, u_0) \in W_3 \times (W_3)_{\text{div}} \times C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3 | \nabla \cdot u_0 = r_2(0)\}. \]  
(132)
Then \( \Phi \) is a map from \( \mathcal{X} \) to \( \mathcal{Z} \). Let
\[ \mathcal{Y} := \Phi(\mathcal{X}). \]  
(133)
Let \( p = (u, P) \in \mathcal{X} \) be an arbitrary point.

Lemma 33. \(d\Phi(p)\) is a topological isomorphism.

Proof. Note that \( \Phi \) is a \(C^1\)-map from \( \mathcal{X} \) to \( \mathcal{Z} \) and from Lemma 32
\[ d\Phi(p) : (h, \beta) \mapsto \begin{bmatrix} \dot{h} + (u \cdot \nabla) h + (h \cdot \nabla) u - \nu \Delta h + \beta \\ \nabla \cdot h \\ h(0) \end{bmatrix} \]  
(134)
is a linear isomorphism and by the open mapping principle a topological linear isomorphism from

\[ T_p\mathcal{X} = \{(h, \beta) \in W_3 \times (W_1)_\circ \} \tag{135} \]

to

\[ T_{\Phi(p)}\mathcal{Y} = \{(a, b, c) \in W_3 \times (W_3)_{\text{div}} \times C^\infty(\mathbb{R}^3/\mathbb{Z}^3)^3 \mid \nabla \cdot c = b(0)\}. \tag{136} \]

The assertion follows. \(\Box\)

**Lemma 34.** \(\mathcal{Y}\) is a Fréchet manifold.

**Proof.** By Lemma 33 \(d\Phi(p)\) is a topological isomorphism so that, since \(p\) is arbitrary, by Lemma 30 \(\Phi : \mathcal{X} \to \mathcal{Y}\) is an open map. Since \(\mathcal{Y} = \Phi(\mathcal{X})\) it is a Fréchet manifold. The assertion follows. \(\Box\)

Let

\[ \varphi_p : \mathcal{X} \xrightarrow{\sim} T_p\mathcal{X} \tag{137} \]

and

\[ \psi_{\Phi(p)} : \mathcal{Y} \xrightarrow{\sim} T_{\Phi(p)}\mathcal{Y} \tag{138} \]

be canonical isomorphisms. Let

\[ U_p := \varphi_p^{-1}(T_p\mathcal{X})(= \mathcal{X}) \tag{139} \]

and

\[ V_{\Phi(p)} := \psi_{\Phi(p)}^{-1}(T_{\Phi(p)}\mathcal{Y}). \tag{140} \]

Observe that by Lemma 33

\[ d\Phi(p) : T_p\mathcal{X} \xrightarrow{\sim} T_{\Phi(p)}\mathcal{Y} \tag{141} \]

is a topological linear isomorphism. Let \(k \in \mathbb{N}\). Let

\[ (W_3^{k,k+2})_{\text{div}} := \{v \mid v = \nabla \cdot w \ (\exists w \in W_3^{k,k+2})\}. \tag{142} \]

Introduce to \((W_3^{k,k+2})_{\text{div}}\) the relative topology as a subset of \(W_1^{k,k+1}\) and to \(\mathcal{Y}\) the relative topology as a subset of

\[ W_3^{k,k} \times (W_3^{k,k+2})_{\text{div}} \times H^{k+2}(\mathbb{R}^3/\mathbb{Z}^3)^3, \tag{143} \]

which defines a seminorm \(p_k\) (it is in fact a norm) of \(\mathcal{Y}\). Let the system of open sets of the topology of \(\mathcal{Y}\) be

\[ \{O\}. \tag{144} \]
Introduce to $T_{\Phi(p)}\mathcal{Y}$ the topology induced from $\theta_{\Phi(p)} := (\psi_{\Phi(p)})^{-1}|_{T_{\Phi(p)}\mathcal{Y}}$, i.e.

$$\{\theta_{\Phi(p)}^{-1}(\mathcal{O})\},$$

(145)

to $T_p\mathcal{X}$ the topology induced from $\theta_{\Phi(p)} \circ d\Phi(p)$, i.e.

$$\{(d\Phi(p))^{-1} \circ \theta_{\Phi(p)}^{-1}(\mathcal{O})\},$$

(146)

to $U_p$ the topology induced from $\theta_{\Phi(p)} \circ d\Phi(p) \circ \varphi_p|_{U_p}$, i.e.

$$\{(\varphi_p|_{U_p})^{-1} \circ (d\Phi(p))^{-1} \circ \theta_{\Phi(p)}^{-1}(\mathcal{O})\},$$

(147)

and to $\mathcal{V}_{\Phi(p)}$ the topology induced from $\theta_{\Phi(p)} \circ \psi_{\Phi(p)}|_{\mathcal{V}_{\Phi(p)}}$, i.e.

$$\{(\psi_{\Phi(p)}|_{\mathcal{V}_{\Phi(p)}})^{-1} \circ \theta_{\Phi(p)}^{-1}(\mathcal{O})\}.$$  

(148)

**Lemma 35.** $U_p = \mathcal{X}$ and $\mathcal{V}_{\Phi(p)} = \mathcal{X}$.

**Proof.** From the definitions, using the canonical isomorphisms (137) and (138), the assertion follows. \qed

**Remark 36.** $\Phi$ satisfies Navier-Stokes condition 1. That $p_k$ is a norm is Navier-Stokes condition 2. (See Remark 65.)

By Lemma 35 $U_p = \mathcal{X}$ and $\mathcal{V}_{\Phi(p)} = \mathcal{X}$. The above topologies are induced by a seminorm $p_k$ such that $p_k(r)$ implies $r = 0$ (for $r \in \mathcal{X}$). Also $U_p$ and $\mathcal{V}_{\Phi(p)}$ are equipped with the ordinary topologies as $\mathcal{X}$ and $\mathcal{X}$. Then $q \mapsto d\Phi(q)$ is continuous with respect to the ordinary topology of $U_p$ and the topology of $T_{\Phi(p)}\mathcal{V}_{\Phi(p)} (= T_{\Phi(p)}\mathcal{Y})$ induced by $p_k$.

**Definition 37.** Let $Y$ be a Banach manifold and $B$ a Banach space. Let $L(q) : T_p Y \to B \ (q \in Y)$ be a map. Then the set of accumulation points of the sequences

$$\sum_i L(\gamma(\xi^j_i))\gamma(\xi^j_i)|E^j_i|,$$

(149)

as $j \to \infty$, where $\gamma : [0,1] \to Y$ runs over all smooth paths from $p \in Y$ to $q \in Y$, $\{E^j_i\}$ all sequences of measurable sets of $[0,1]$ such that $\prod_i E^j_i = [0,1]$, and $\sup_i |E^j_i| \to 0 (j \to \infty)$ and $\xi^j_i$ all elements of $E^j_i$, if exists, is denoted by $\int_p^q L(q')$.

**Remark 38.** In general $q \mapsto \int_p^q L(q')$ is a multi-valued map.

The following is a result of [17], of which expression is slightly changed.
Theorem 39 (M. A. Rieffel). Let \((X, S, \mu)\) be a \(\sigma\)-finite positive measure space and let \(B\) be a Banach space. Let \(m\) be a \(B\)-valued measure on \(S\). Then \(m\) is the indefinite integral with respect to \(\mu\) of a \(B\)-valued Bochner integrable function on \(X\) if and only if

1. \(m(E) = 0\) whenever \(\mu(E) = 0\), \(E \subseteq S\),
2. the total variation \(|m|\) of \(m\) is a finite measure,
3. given \(E \subseteq S\) with \(0 < \mu(E) < \infty\) there exists an \(F \subseteq E\) such that \(\mu(F) > 0\) and

\[
A_F(m) := \{ \frac{m(F')}{\mu(F')} \mid F' \subseteq F, \mu(F') > 0 \}
\]

is relatively (norm) compact.

Definition 40. A topological manifold \(Y\) is a manifold with a topological linear space \(T_qY\) for each \(q \in Y\) such that there exists for each \(q \in Y\) a homeomorphism \(\theta_q\) from a neighbourhood \(W\) of \(q\) to a neighbourhood \(W'\) of \(0 \in T_qY\). A topological manifold \(Y\) is parametrized if for each \(q \in Y\) there exists a fundamental system \(\{W'q\}_\tau\) of neighbourhoods of \(0 \in T_qY\) decreasing as \(\tau \to 0\). A \(C^1\)-map between parametrized topological manifolds \(Y_1, Y_2\) with fundamental systems \(\{W'_1q\}_\tau, \{W'_2q\}_\tau\) of neighbourhoods is a map \(\xi\) with the following property: for each \(q \in Y_1\) there exists \(d\xi(q)\) such that for small \(\tau\) and for \(q' - q \in W'_1q\)

\[
\xi(q') - \xi(q) - d\xi(q)(q' - q) \in W'_2q_{\alpha(\tau)},
\]

where a specified neighbourhood \(W_1\) of \(q \in Y_1\) and a specified neighbourhood \(W'_1\) of \(0 \in T_qY_1\) are identified. \(d\xi(q)\) is the Fréchet derivative of \(\xi\) at \(q\). A (not necessarily \(C^1\)) map \(\xi : Y_1 \to Y_2\) has derivative a.e. along a smooth path \(\gamma\) if \(d(\xi(\gamma(t)))(t \in [0, 1])\) exists a.e. for a smooth path \(\gamma : [0, 1] \to Y_1\).

The following is a consequence of [5], Chapter 5, Section 5.7, Theorem 1 (Rellich-Kondrachov Compactness Theorem).

Theorem 41. Let \(k \in \mathbb{Z}_{\geq 0}\). \(\Omega \subseteq \mathbb{R}^3\) be a bounded open domain. Let \(H^k(\Omega)\) be the Sobolev space. Then the inclusion \(H^{k+1}(\Omega) \hookrightarrow H^k(\Omega)\) is a compact operator.

Lemma 42. Let \(\mathcal{X}_p\) be a sufficiently small convex neighbourhood of \(p \in \mathcal{X}\) identified with the corresponding neighbourhood of \(0 \in T_q\mathcal{X}_p\) (\(q \in \mathcal{X}_p\)). Let \(\{q_j\} \subseteq \mathcal{X}_p\) be a sequence. Then there exists a subsequence of \(\{q_j\}\) convergent to \(0 \in \mathcal{X}_p\) with respect to the ordinary topology.

Proof. Let \(k_1, k_2 \in \mathbb{Z}_{\geq 0}\). Let \((u_j, P_j) := q_j\). By Theorem 41 possibly passing to a subsequence \((\partial_{k_1}^{k_1} u_j(t), \partial_{k_1}^{k_1} P_j(t))\) is convergent for fixed \(t \in I\). Let

\[
(\partial_{k_1}^{k_1} u(t), \partial_{k_1}^{k_1} P(t)) := \lim_{j \to \infty} (\partial_{k_1}^{k_1} u_j(t), \partial_{k_1}^{k_1} P_j(t))
\]

By diagonal argument the same formula holds for any \(t_0 \in I',\) where \(I'\) is a dense countable subset of \(I\). By assumption \(||(\partial_{k_1}^{k_1} u_j, \partial_{k_1}^{k_1} P_j)||_{W^{k\ast}_{3,\

\}}

\]

23
$(|\alpha| \leq k_2)$ is bounded on $I$ so that $(\partial_t^{k_1} \partial_x^\alpha u(t_0), \partial_t^{k_1} \partial_x^\alpha \nabla p(t_0))$ ($t_0 \in I'$) extends to a continuous function

$$
(\lim_{t_0 \to t} \partial_t^{k_1} \partial_x^\alpha u(t_0), \lim_{t_0 \to t} \partial_t^{k_1} \partial_x^\alpha P(t_0)),
$$

(153)

where $t \in I$. The resulting function is denoted by the same symbol. Since $||((\partial_t^{k_1+1} \partial_x u_j, \partial_t^{k_1+1} \partial_x^\alpha P_j))||_{W_a^{0,a} \times (W_a^a)}$ is bounded there exists $M > 0$ such that

$$
||((\partial_t^{k_1} u_j(t), \partial_t^{k_1} P_j(t)) - (\partial_t^{k_1} u_j(t_0), \partial_t^{k_1} P_j(t_0)))||_{W_a^{0,k_2} \times (W_a^a)}
$$

(154)

$$
\leq M|t - t_0|
$$

(155)

and

$$
||((\partial_t^{k_1} u(t), \partial_t^{k_1} P(t)) - (\partial_t^{k_1} u(t_0), \partial_t^{k_1} P(t_0)))||_{W_a^{0,k_2} \times (W_a^a)}
$$

(156)

$$
\leq M|t - t_0|.
$$

(157)

For any $t \in I$ and any $\epsilon > 0$ there exists $t_0 \in A$ such that $M|t - t_0| < \epsilon$ so that by above for large $j$

$$
||((\partial_t^{k_1} u_j(t), \partial_t^{k_1} P_j(t)) - (\partial_t^{k_1} u(t), \partial_t^{k_1} P(t)))||_{W_a^{0,k_2} \times (W_a^a)}
$$

(158)

$$
\leq ||((\partial_t^{k_1} u_j(t), \partial_t^{k_1} P_j(t)) - (\partial_t^{k_1} u_j(t_0), \partial_t^{k_1} P_j(t_0)))||_{W_a^{0,k_2} \times (W_a^a)}
$$

(159)

$$
+ ||((\partial_t^{k_1} u_j(t_0), \partial_t^{k_1} P_j(t_0)) - (\partial_t^{k_1} u(t), \partial_t^{k_1} P(t)))||_{W_a^{0,k_2} \times (W_a^a)}
$$

(160)

$$
+ ||((\partial_t^{k_1} u_j(t_0), \partial_t^{k_1} P_j(t_0)) - (\partial_t^{k_1} u(t), \partial_t^{k_1} P(t)))||_{W_a^{0,k_2} \times (W_a^a)}
$$

(161)

$$
< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
$$

(162)

Thus possibly passing to a subsequence $(\partial_t^{k_1} u_j(t), \partial_t^{k_1} P_j(t)) \to (\partial_t^{k_1} u(t), \partial_t^{k_1} P(t))$ ($t \in I$). Since $\mathbb{R}^3 / \mathbb{Z}^3$ is compact the above formula also shows that the convergence is uniform. Since $k_1, k_2$ are arbitrary the assertion follows.

The set $X$ equipped with the topology induced by $p_k$ is, if exists, denoted by $X^w$.

**Lemma 43.** Let $\mathcal{X}_p$ be a sufficiently small convex neighbourhood of $p \in U_p$ with respect to the ordinary topology identified with the corresponding neighbourhood of $0 \in T_q U_p (= T_q \mathcal{X}_p)$ ($q \in \mathcal{X}_p$). Then for any sequence $\{q_j\} \subset \mathcal{X}_p^w$ convergent to $0 \in \mathcal{X}_p^w$, it is convergent to $0 \in \mathcal{X}_p$ with respect to the ordinary topology.

**Proof.** Observe that $C^\infty(\mathbb{R}^3 / \mathbb{Z}^3) = H^\infty(\mathbb{R}^3 / \mathbb{Z}^3)$. Assume possibly after passing to a subsequence that $q_j \notin c \mathcal{X}_p$ ($0 < c < 1$) for large $j$. $\mathcal{X}_p$ is sufficiently small so that by Lemma 42 there exists a subsequence $\{q_{j_k}\}$ convergent with respect to the ordinary topology. Let $W^1, \ldots, W^m$ be neighbourhoods of $p \in U_p$ with respect to the topology induced by $p_k$ and $W^1, \ldots, W^m$ their closures with
respect to this topology. Let \( \mathcal{S}_p \) be the closure of \( \mathcal{S} \) with respect to the ordinary topology. Then since \( q_j \in W^1 \cap \ldots \cap W^m \cap (\mathcal{S}_p \cap c\mathcal{S}_p) \) for large \( l \) and \( W^1, \ldots, W^m \) are arbitrary and since from above \( (\mathcal{S}_p \cap c\mathcal{S}_p) \) is compact with respect to the ordinary topology and so the topology induced by \( p_k \) an elementary argument shows that

\[
\bigcap_{W}(W \setminus c\mathcal{S}_p) \neq \emptyset. \tag{163}
\]

This contradicts with the definition of the topology of \( \mathcal{U}_p \) induced by \( p_k \). Thus \( q_j \in c\mathcal{S}_p \) for large \( j \). Since \( c \) is arbitrary \( \{q_j\} \subset \mathcal{S}_p \) is convergent to \( 0 \in \mathcal{S}_p \) with respect to the ordinary topology. The assertion follows.

The integral with respect to the topologies induced by \( p_k \) (resp. with respect to the ordinary topology of \( \mathcal{U}_p \) and the topology of \( \mathcal{V}_{\Phi(p)} \) induced by \( p_k \)) is denoted by \( \int \) (resp. \( \int^\prime \)). See Definition 37.

**Lemma 44.** There exist a sufficiently small convex neighbourhood \( \mathcal{S}_p \) of \( p \in \mathcal{U}_p \) with respect to the ordinary topology and a sufficiently small neighbourhood of \( 0 \in (T_{\Phi(p)} \mathcal{S})^w \) identified with a sufficiently small neighbourhood \( \mathcal{V}_{\Phi(p)} \) of \( \Phi(p) \in \mathcal{V}_{\Phi(p)}^w \) such that the multi-valued map \( q \in \mathcal{S}_p \rightarrow \int_0^1 d\Phi(q') \in \mathcal{V}_{\Phi(p)}^w \) has a continuous branch.

**Proof.** Since \( \mathcal{S}_p \) is sufficiently small the map \( d\Phi(q) \) \( (q \in \mathcal{S}_p) \) from \( T_{\Phi(p)} \mathcal{S}_p \) to \( \mathcal{V}_{\Phi(p)}^w \) is continuous and the function \( q \mapsto \left( \int_0^1 s d\Phi(q') \right) \| (T_{\Phi(p)} \mathcal{S})^w \) is well-defined. Let \( \gamma_q(t) := (1 - t)p + tq \) \( (q \in \mathcal{S}_p) \). Since \( d\Phi \) is well-defined and a continuous map from \( \mathcal{S}_p \) to \( \mathcal{V}_{\Phi(p)}^w \) there exists \( M > 0 \) such that

\[
\lim_{h \to 0} \frac{1}{h} \left( \int_{\gamma_q(t)}^{\gamma_q(t+1+h)} d\Phi(q') \right) \| (T_{\Phi(p)} \mathcal{S})^w = 1 \tag{164}
\]

\[
= \lim_{h \to 0} \frac{1}{h} \left( \int_0^1 s d\Phi((1 - t)\gamma_q(t_1) + t\gamma_q(t_1 + h)) (\gamma_q(t_1 + h) - \gamma_q(t_1)) dt \right) \| (T_{\Phi(p)} \mathcal{S})^w = 1 \tag{165}
\]

\[
= \lim_{h \to 0} \left( \int_0^1 s d\Phi((1 - t)\gamma_q(t_1) + t\gamma_q(t_1 + h)) (q - p) dt \right) \| (T_{\Phi(p)} \mathcal{S})^w = 1 \tag{166}
\]

\[
\leq M. \tag{167}
\]

By Theorem 39 there exists an integrable function \( S_{\gamma_q} \) (see Definition 37) such that

\[
\left( \int_{\gamma_q(t_1)}^{\gamma_q(t_2)} s d\Phi(q') \right) = \int_{\gamma_q(t_1)}^{\gamma_q(t_2)} S_{\gamma_q} dt, \tag{168}
\]

where \( 0 \leq t_1 \leq t_2 \leq 1 \). Then by Lemma 43 an easy argument shows that the map \( q \in \mathcal{S}_p \rightarrow \int_0^1 S_{\gamma_q} dt \in \mathcal{V}_{\Phi(p)}^w \) gives a desired branch. The assertion follows.
Observe that, where $\{W r\}_r$ is a fundamental system of neighbourhoods of $0 \in U_p$ with respect to the topology induced by $p_k$, a fundamental system of $0 \in X_p^w$ is given by $\{W_r \cap X_p^w\}_r$.

**Lemma 45.** There exist a sufficiently small convex neighbourhood $X_p$ of $p \in U_p$ with respect to the ordinary topology and a sufficiently small neighbourhood of $0 \in (T_{\Phi(p)}Y)^w$ identified with a sufficiently small neighbourhood $Y_{\Phi(p)}^w$ of $\Phi(p) \in V_{\Phi(p)}^w$ such that

$$\Phi(q) = \Phi(p) + \int_p^q d\Phi(q') \quad (q \in X_p^w, \Phi(q) \in Y_{\Phi(p)}^w)$$

holds for a well-defined continuous branch of $q \in X_p^w \mapsto \int_p^q d\Phi(q') \in Y_{\Phi(p)}^w$.

**Proof.** By Lemma 44 the multi-valued map $q \in X_p^w \mapsto \int_p^q d\Phi(q') \in Y_{\Phi(p)}^w$ has a continuous branch so that the function

$$\Phi'(q) := \Phi(p) + \int_p^q d\Phi(q')$$

has a well-defined continuous branch. Note that since $\Phi'$ has the derivative a.e. along $\gamma_q$ and since the topology of $X_p^w$ induced by $p_k$ is weaker than the ordinary one it is obtained that (*) the derivatives a.e. along $\gamma_q$ of $\Phi'$ with respect to the both topologies at $q$ are equal to $d\Phi(q)$ a.e. in the path. On the other hand it is true that

$$\Phi(q) = \Phi(p) + \left( \int_p^q \right) d\Phi(q').$$

Then $\Phi$ and $\Phi'$ satisfy the same conditions: (i) $\Xi(p) = \Phi(p)$ and (ii) the derivative of $\Xi(q)$ a.e. along $\gamma_q$ with respect to the ordinary topology of $X_p$ and the topology of $Y_{\Phi(p)}^w$ induced by $p_k$ coincides with $d\Phi(q)$ a.e. in the path. It follows that $\Phi(q) = \Phi'(q)$. By (*) it is obtained for $t \in [0, 1]$ that

$$S_{\gamma_q}(t) = \frac{d\Phi(\gamma_q(t))}{d\Phi(q'(t))}$$

(if LHS exists and is equal to $d(\Phi'(\gamma_q(t)))$). Thus for this branch

$$\Phi(q) = \Phi(p) + \int_p^q d\Phi(q').$$

The assertion follows. □

$X_p$ and $X_p^2$ are identified with the corresponding neighbourhoods of $0 \in T_q X_p$ ($q \in X_p$) and $0 \in T Q T_q X_p$ ($Q \in X_p^2$).

**Lemma 46.** Shrink $X_p$ if necessary. Then $d\Phi : (X_p^2)^w \to (T_{\Phi(p)}Y)^w$ is continuous and the Fréchet derivative of $\Phi : X_p^w \to Y_{\Phi(p)}^w$ (see Definition 40) is equal to $d\Phi$ on $(X_p^2)^w$. 

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Proof. As in Lemma 44 and Lemma 45 replacing $\Phi$ with $d\Phi$ it is proved that $d\Phi$ is continuous on $(\mathcal{X}^2_p)^w$. From this and Lemma 45 the assertion follows easily. \hfill $\square$

Lemma 47. Shrink $\mathcal{X}_p$ if necessary. Then $d(d\Phi): (\mathcal{X}^2_p)^w \rightarrow (TT\mathcal{Y}_{\Phi(p)})^w$ is continuous and the Fréchet derivative of $d\Phi: (\mathcal{X}^2_p)^w \rightarrow (TT\mathcal{Y}_{\Phi(p)})^w$ (see Definition 40) is equal to $d(d\Phi)$ on $(\mathcal{X}^2_p)^w$.

Proof. This is proved as in Lemma 46. \hfill $\square$

Lemma 48. Let $L((T_p\mathcal{X}_p)^w, (T_{\Phi(p)}\mathcal{Y}_{\Phi(p)})^w)$ be the set of continuous linear operators from $(T_p\mathcal{X}_p)^w$ to $(T_{\Phi(p)}\mathcal{Y}_{\Phi(p)})^w$. Then there exists $M' > 0$ such that

$$
||d\Phi(q') - d\Phi(q)||_{L((T_p\mathcal{X}_p)^w, (T_{\Phi(p)}\mathcal{Y}_{\Phi(p)})^w)} \leq M'||q' - q||_{\mathcal{X}_p^w},
$$

for $q, q', q' - q \in \mathcal{X}_p^w$, and such that

$$
||\Phi(q') - \Phi(q) - d\Phi(q)(q' - q)||_{\mathcal{Y}_p^w} \leq M'||q' - q||^2_{\mathcal{X}_p^w},
$$

for $q, q', q' - q \in \mathcal{X}_p^w$.

Proof. By Lemma 47 $d(d\Phi): (\mathcal{X}^2_p)^w \rightarrow (TT\mathcal{Y}_{\Phi(p)})^w$ is continuous. Shrinking $\mathcal{X}_p$ if necessary there exists $M' > 0$ such that for $Q \in (\mathcal{X}^2_p)^w$

$$
||d(d\Phi)(Q)|| \leq M'.
$$

Let $\gamma(t) := (1-t)q + tq'$ ($t \in [0, 1]$). Then, by Lemma 47

$$
||d\Phi(q') - d\Phi(q)||_{L((T_p\mathcal{X}_p)^w, (T_{\Phi(p)}\mathcal{Y}_{\Phi(p)})^w)} = \left(\int_0^1 \frac{\partial}{\partial t} d\Phi(\gamma(t))dt\right) ||\gamma(t)||_{\mathcal{X}_p^w} \leq M'||q' - q||_{\mathcal{X}_p^w},
$$

and

$$
||\Phi(q') - \Phi(q) - d\Phi(q)(q' - q)||_{\mathcal{Y}_p^w} = \left(\int_0^1 \frac{\partial^2}{\partial t^2} \Phi(\gamma(t))dt\right) ||\gamma(t)||_{\mathcal{Y}_p^w} \leq M'||q' - q||^2_{\mathcal{X}_p^w}.
$$

The assertion follows. \hfill $\square$

Take the completions $\mathcal{U}_p^k$ and $\mathcal{V}_p^k$ of $\mathcal{X}_p^w$ and $\mathcal{Y}_{\Phi(p)}^w$ induced from $p_k$. By Lemma 48, shrinking $\mathcal{U}_p^k$ and $\mathcal{V}_\Phi^k$ if necessary, $d\Phi$ extends to a continuous map, $\Phi$ extends to a $C^1$-map from $\mathcal{U}_p^k$ to $\mathcal{V}_{\Phi(p)}^k$ and the Fréchet derivative with respect to the topologies of $\mathcal{U}_p^k$ and $\mathcal{V}_{\Phi(p)}^k$ of the extended $\Phi$ at $q$ is equal to the value
\(d\Phi(q)\) at \(q\) of the extended \(d\Phi\). \(d\Phi(p)\) extends to a topological isomorphism between the completions \(T_p\mathscr{U}^k_p\) and \(T_{\Phi(p)}V^k_{\Phi(p)}\) of \((T_p\mathscr{X})^w\) and \((T_{\Phi(p)}\mathscr{Y})^w\) induced from \(p_k\) so that since \(\mathscr{X}_p\) is sufficiently small \(d\Phi(q)\) \((q \in \mathcal{U}^k_p)\) is a topological isomorphism. By Theorem 12 there exist sufficiently small neighbourhoods \(U_p\) and \(V_{\Phi(p)}\) of \(p \in \mathcal{U}^k_p\) and \(\Phi(p) \in V^k_{\Phi(p)}\) such that the extended
\[
\Phi : U_p \xrightarrow{\sim} V_{\Phi(p)}
\]
is an isomorphism.

**Remark 49.** We have taken \(O = \mathcal{U}^k_p\), \(X\) the linear hull of \(\mathcal{U}^k_p\), \(Y\) the linear hull of \(V^k_{\Phi(p)}\), \(\xi = \Phi\) and \(a_0 = p\).

Observe that \(V_{\Phi(p)}\) is an open set of the completion of \(\mathscr{Y}\). We thus obtain:

**Lemma 50.** \(V_{\Phi(p)}\) \((p \in \mathscr{X})\) form a Banach manifold.

Replacing \(I\) with an interval \(J := [t_0, t_0 + T_0]\) \((0 \leq t_0 < T\) and \(T_0 > 0\) is small) we define \(\mathscr{X}^J\), \(\mathscr{Y}^J\), \(U^J_p\), \(\Phi^J\) etc. in the same way as \(\mathscr{X}\), \(\mathscr{Y}\), \(U_p\), \(\Phi\) etc. The above two (1 and 2) of Navier-Stokes conditions also hold if we replace \(I\) with \(J\) (see Remark 65).

**Remark 51.** Navier-Stokes condition 3 (see Remark 65) is satisfied.

We need a lemma.

**Lemma 52.** If \(U^J_{p_1} \cap U^J_{p_2} \neq \emptyset\), it is open in \(U^J_{p_1}\).

**Proof.** Let \(p \in U^J_{p_1} \cap U^J_{p_2} \cap \mathscr{X}^J\). The tangent space \(T_pU^J_{p_1}\) of \(U^J_{p_1}\) at \(p\) and the tangent space \(T_pU^J_{p_2}\) of \(U^J_{p_2}\) at \(p\) are by definition given as the completions of \(T_p\mathscr{X}^J\) and \(\Phi^J\) is a local diffeomorphism on \(U^J_{p_1}\) and \(U^J_{p_2}\). Thus \(T_pU^J_{p_1} = T_pU^J_{p_2} = (d\Phi(p))^{-1}(\text{completion of } T_{\Phi(p)}\mathscr{Y}^J)\), where the completion of \(T_{\Phi(p)}\mathscr{Y}^J\) is induced from \(\mathscr{Y}^J\) \(\subset \bigcup_p V^J_{\Phi(p)}\). There exists a canonical isomorphism \(\varphi^J_{p_1}|_{W}\)
from a neighbourhood \(W\) of \(p \in \mathscr{X}^J\) to the corresponding neighbourhood \(W^1\) of \(0 \in T_p\mathscr{X}^J\) and \(\mathcal{U}^k_{p_1} \cup \mathcal{U}^k_{p_2}\) is the completion of \(\mathscr{Y}^J\) induced from \(p_k\). Extend \(\varphi^J_{p_1}|_{W}\) to \(\bar{W}^1\) (resp. \(\bar{W}^2\), where \(W^1\) (resp. \(W^2\)) is the closure of \(W\) in \(U^J_{p_1}\) (resp. \(U^J_{p_2}\)), to obtain a homeomorphism \(\varphi^J_{p_1,1}\) (resp. \(\varphi^J_{p_2,2}\)). By definition \(\varphi^J_{p_1,1}(W^1)\) (resp. \(\varphi^J_{p_2,2}(W^2)\)) is the closure of \(W^1\) in \(T_pU^J_{p_1}\) (resp. \(T_pU^J_{p_2}\)). Since \(T_pU^J_{p_1} = T_pU^J_{p_2}\) it follows that \(\varphi^J_{p_1,1}(W^1) = \varphi^J_{p_2,2}(W^2)\) and since \(U^J_{p_1} \cap U^J_{p_2} \subset \mathcal{U}^k_{p_1} \cup \mathcal{U}^k_{p_2}\), from the above, an easy argument shows that \((\varphi^J_{p_1,1})^{-1}((\bar{W}^2)c)\) is open in \(U^J_{p_1}\). By the definition of the topology \(U^J_{p_1} \cap U^J_{p_2} \cap \mathscr{X}^J\) is dense in \(U^J_{p_1} \cap U^J_{p_2}\).

Let \(p' \in U^J_{p_1} \cap U^J_{p_2}\) be an inner point of \(U^J_{p_2}\). Take a small neighbourhood \(N\) of \(p'\) in \(U^J_{p_2}\) and consider \(U^J_{p_1} \cap N \cap \mathscr{X}^J\), which is by the same argument as above open in \(U^J_{p_1} \cap \mathscr{X}^J\), that is, of the form \(O \cap U^J_{p_1} \cap \mathscr{X}^J\), where \(O\) is an open set.
in $U_{p_1}^J$. Take the closure of $U_{p_1}^J \cap N \cap \mathcal{X}^J$. Then the set of all inner points of this closure is open in $U_{p_1}^J$, and the boundary of the closure does not contain $p'$. Hence $U_{p_1}^J \cap U_{p_2}^J$ is open in $U_{p_1}^J$. The assertion follows.

\[ \square \]

**Corollary 53.** $U_p^J (p \in \mathcal{X}^J)$ form a Banach manifold.

We shall prove the local existence and uniqueness of a sufficiently smooth solution of equation (7) (of which smoothness depends on $k \in \mathbb{N}$).

Let $t_0 \in I$. Introduce to the inductive limits $\mathcal{U}_{t_0}$ of $\bigcup_p U_p^J$ for $J \ni t_0$ and $\mathcal{C}_{t_0}$ of $\bigcup_p V_p^J$ for $J \ni t_0$. Choose $W^k, k = 1, 2, 3$.

\[ \mathcal{X}^J := \{(f, r_2, u_0) \in (W^k)^J \times (\{(W^{k+2})_d\}^J \times H^{k+2}(\mathbb{R}^3/\mathbb{Z}^3)^3 \quad (184) \]

\[ \mathcal{J} := \{(f, r_2, u_0) \in \mathcal{X}^J \mid \nabla \cdot u_0 = r_2(t_0)\} \quad (185) \]

for $J \ni t_0$ the natural topologies (the quotient topologies induced from the maps $\prod_p U_p^J \to \mathcal{U}_{t_0}$ and $\prod_p \mathcal{J} \to \mathcal{C}_{t_0}$). We first prove the following lemma.

**Lemma 54.** Let $r := (f, r_2, u_0) \in \mathcal{X}^J$. There exists $(u, \nabla p) \in \mathcal{X}^J$ such that

\[ \begin{align*}
\left( \begin{array}{l}
(\dot{u} + (u \cdot \nabla)u - \nu \Delta u + \nabla p)|_{t=t_0} = f(t_0), \\
(\nabla \cdot u)|_{t=t_0} = r_2(t_0), \\
u|_{t=t_0} = u_0, \\
(\nabla \cdot \dot{u})|_{t=t_0} = \dot{r}_2(t_0).
\end{array} \right. \quad (186) \]

Proof. Since

\[ (\dot{u} + (u \cdot \nabla)u - \nu \Delta u + \nabla p)|_{t=t_0} = f(t_0), \quad (187) \]

by assumption

\[ -\dot{u}|_{t=t_0} - \nabla p(t_0) = (u_0 \cdot \nabla)u_0 - \nu \Delta u_0 - f(t_0). \quad (188) \]

By $(\nabla \cdot \dot{u})|_{t=t_0} = \dot{r}_2(t_0)$, $-\dot{u}|_{t=t_0}$ is determined up to $\text{Ker}(\nabla \cdot (\cdot))$. Then by Theorem 17 it is confirmed that there exists such $(u, \nabla p)$. The assertion follows.

\[ \square \]

$\{\Phi^J\}_J$ induces a map $\Phi_{t_0} : \mathcal{U}_{t_0} \to \mathcal{C}_{t_0}$.

**Lemma 55.** The induced map $\Phi_{t_0} : \mathcal{U}_{t_0} \to \mathcal{C}_{t_0}$ is a homeomorphism.

Proof. $\Phi^J : \bigcup_p U_p^J \to \bigcup_p V_p^J$ is a local diffeomorphism so that by definition $\Phi_{t_0} : \mathcal{U}_{t_0} \to \mathcal{C}_{t_0}$ is a local homeomorphism. $\mathcal{U}_{t_0}$ is connected (since $\mathcal{X}^J$ is connected and each connected component of $U_p^J$ intersects with $\mathcal{X}^J$) and $\mathcal{C}_{t_0}$ is simply connected. We claim that $\Phi_{t_0}$ is surjective. Then $\Phi_{t_0}$ is a homeomorphism. By Lemma 54 it is proved that for any $r := (f, r_2, u_0) \in \mathcal{X}^J$ there
exists a function \( q' \in \bigcup U^J_p \) such that \( \Phi^J(q')|_{t=t_0} = r(t_0) \), where \( r^*_t = r^*(t) := (f^*(t), r^*_2(t), u^*_n) \) for \( r^* := (f^*, r^*_2, u^*_n) \in \mathcal{P}^J \). Since \( \Phi^J \) is an open map, there exists a deformation \( q \in \bigcup U^J_p \) of \( q' \) such that \( \Phi^J(q)|_t = r(t) \) on a neighbourhood of \( t_0 \). More precisely there exists sufficiently small \( \epsilon > 0 \) such that any \( r^* \in \mathcal{P}^J \) with
\[
\|d(r^*, \Phi^J(q'))_{W^J_2 \times (W^J_2)_\text{div} \times C^\infty(\mathbb{R}^3 / \mathbb{Z}^3)}\| < \epsilon
\] (distance) is in \( \text{Im} \Phi^J \). Let
\[
(C^\infty(\mathbb{R}^3 / \mathbb{Z}^3)^3)_\text{div} := \{ v \mid v = \nabla \cdot w \ (\exists w \in C^\infty(\mathbb{R}^3 / \mathbb{Z}^3)^3) \}. \tag{190}
\]
It is obtained that for small \( t_1 > 0 \),
\[
\|d(r(t), \Phi^J(q'))_{C^\infty(\mathbb{R}^3 / \mathbb{Z}^3)^3 \times (C^\infty(\mathbb{R}^3 / \mathbb{Z}^3))^3} \| < \epsilon \ (t_0 \leq t < t_0 + t_1) \tag{191}
\]
(distance). By formula (189) and formula (191) there exists \( q \in \bigcup U^J_p \) such that \( \Phi^J(q)|_{t} = r(t) \) \((0 \leq t < t_0 + t_1)\). So the image of \( \Phi|_{t_0} \) contains the inductive limit \( \mathcal{P}^J \) for \( J \ni t_0 \). Observe that the completion of \( \mathcal{P}^J \) in \( \mathcal{P}^{(k),J} \) coincides with the latter space so that the completion of \( \mathcal{C}_{t_0}^\infty \) in \( \mathcal{C}_{t_0} \) does with \( \mathcal{C}_{t_0} \). From these the assertion is confirmed. Thus \( \Phi|_{t_0} \) is a homeomorphism. \( \square \)

**Remark 56. Navier-Stokes condition 4 (see Remark 65) is satisfied.**

We obtain a bijection \( \Phi : \mathcal{U} := \coprod_{t_0} \mathcal{U}_{t_0} \rightarrow \mathcal{C} := \coprod_{t_0} \mathcal{C}_{t_0} \) and introducing a sheaf structure to \( \mathcal{C} \) induced from \( \mathcal{U} \) through this bijection a sheaf isomorphism, where the topology of \( I \) is generated by \( J \)'s. We thus conclude as follows.

**Corollary 57.** \( \Phi : \mathcal{U} \rightarrow \mathcal{C} \) is a sheaf isomorphism.

Let \( (f, r_2, u_0) \in \mathcal{P}^{(k),J} \). Then there exists a section \( \tilde{\rho} := (u, \nabla p) \) of \( \mathcal{U} \) such that \( \Phi(\tilde{\rho}) = (f, r_2, u_0) \) on \( I_1 \), where \( I_1 := [0, T_1) \) \((0 < T_1 \leq T) \) (or \( I_1 := [0, T) \)) is maximal and \( \tilde{\rho} \) is locally unique in this topology of \( I_1 \) because \( \Phi \) is an isomorphism. Hence \( \tilde{\rho} \) is unique. Consider the following equation on \( I_1 \):
\[
\begin{bmatrix}
\frac{\partial u}{\partial t} + (u \cdot \nabla) u - \nu \Delta u + \nabla p \\
\nabla \cdot u \\
u_{n=0}
\end{bmatrix} := \Phi((u, \nabla p)) = \begin{bmatrix} f \\
r_2 \\
u_0
\end{bmatrix}. \tag{192}
\]
We say \((u, \nabla p)\) satisfies equation (7) on \( I_1 \) if it satisfies equation (192). Now we proved the following.

**Theorem 58.** Let \( \nu > 0 \). Let \( k \in \mathbb{N} \). Let \( I := [0, T] \) \((T > 0) \). Let
\[
(f, r_2, u_0) \in W^{3,k}_3 \times (W^{3,k+2}_3)_\text{div} \times H^{k+2}(\mathbb{R}^3 / \mathbb{Z}^3)^3 \tag{193}
\]
such that \( \nabla \cdot u_0 = r_2(0) \). Then there exists an unique \((u, \nabla p)\) satisfying equation (7) on \( I_1 := [0, T_1) \) \((0 < T_1 \leq T) \) (or on \( I_1 := [0, T) \)) and that \( I_1 \) is maximal.
We introduced to \( \mathcal{Y} \) the relative topology induced from (143) (which depends on \( k \)) and defined the Banach manifold \( \bigcup U_p^i \) in Corollary 53 and the sheaf \( \mathcal{Y} \) after the proof of Lemma 55. Now we remark the following:

**Remark 59.** In fact \((u, \nabla p) \in \mathcal{Y}(I_1)\).

We are going to prove Theorem 2. That \( \mathcal{Y} = \mathcal{Z} \) is proved by Navier-Stokes conditions (see Remark 65). Since \((f, 0, u_0) \in A \) and \( \mathcal{Y} = \Phi(\mathcal{X}) \) there exists \( p \in \mathcal{X} \) such that \( \Phi(p) = (f, 0, u_0) \), which defines a smooth solution of equation (7). We formalize this in the following way.

Let \( \Omega \subset \mathbb{R}^3/\mathbb{Z}^3 \). Let

\[
W_{\Omega,n} := C^\infty(I, C^\infty(\Omega)^n), \\
(W_{\Omega,1})_v := \{ v | v = \nabla w \ (\exists w \in W_{\Omega,1}) \}, \\
(W_{\Omega,3})_{div} := \{ v | v = \nabla \cdot w \ (\exists w \in W_{\Omega,3}) \}, \\
\mathcal{X}_{\Omega} := W_{\Omega,3} \times (W_{\Omega,1})_v. 
\]

**Lemma 60.** Let \((f, r_2, u_0) \in \mathcal{Z} \). Let \( x \in \mathbb{R}^3/\mathbb{Z}^3 \). Then there exists \((u^x, \nabla p^x) \in \mathcal{X}_{\{x\}} \) satisfying equation (7) on \( I \times \{x\} \).

**Proof.** The assertion easily follows. \( \square \)

**Lemma 61.** Let \((f, r_2, u_0) \in \mathcal{Z} \). Let \( x \in \mathbb{R}^3/\mathbb{Z}^3 \). Then there exist a compact neighbourhood \( K_x \) of \( x \) and \((u^{K_x}, \nabla p^{K_x}) \in \mathcal{X}_{K_x} \) satisfying equation (7) on \( I \times K_x \).

**Proof.** By Lemma 60 it follows that there exists \((u^x, \nabla p^x) \in \mathcal{X}_{\{x\}} \) satisfying equation (7) on \( I \times \{x\} \). Extend it arbitrarily to \( I \times (\mathbb{R}^3/\mathbb{Z}^3) \) to obtain \((u', \nabla p') \in \mathcal{X} \). Note that

\[
d((f, r_2, u_0), \Phi(u', \nabla p'))_{W_{\{x\},3} \times (W_{\{x\},1})_{div} \times C^\infty(\{z\})^3} \]

(distance) is small for any \( z \) around \( x \). Then since by Lemma 30 \( \Phi : \mathcal{X} \rightarrow \mathcal{Y} \) is an open map there exist a compact neighbourhood \( K_x \) of \( x \) and \((u^{K_x}, \nabla p^{K_x}) \in \mathcal{X}_{K_x} \) satisfying equation (7) on \( I \times K_x \). The assertion follows. \( \square \)

**Remark 62.** Navier-Stokes condition 5 (see Remark 65) is satisfied.

**Lemma 63.** \( \mathcal{Y} = \mathcal{Z} \).

**Proof.** Let \((f, r_2, u_0) \in \mathcal{Z} \). By Lemma 61 it is obtained that there exists a family \( \{(u^{K_x}, \nabla p^{K_x})\}_x \) such that each \((u^{K_x}, \nabla p^{K_x}) \) satisfies equation (7). Since \( \mathbb{R}^3/\mathbb{Z}^3 \) is compact we may obtain a finite set \( \{x_\lambda\} \) such that each \( K_{x_\lambda} \) intersects with another in a set of Lebesgue measure 0 and \( \bigcup_{\lambda} K_{x_\lambda} = \mathbb{R}^3/\mathbb{Z}^3 \). It follows that there exists \((u, \nabla p) \in (L^2(I \times (\mathbb{R}^3/\mathbb{Z}^3))^3)^2 \) that is smooth a.e. and
satisfies equation (7) a.e. By Lemma 30 and Theorem 58 \( \Phi : \mathcal{X} \to \mathcal{Y} \) is a homeomorphism. Extend \( \Phi^{-1} \) to a continuous map from \( \mathcal{Y} (\subset \mathcal{X}) \) to

\[
\{(u, \nabla p) \in (L^2(I \times (\mathbb{R}^3/\mathbb{Z}^3))^3 \mid u, \nabla p \text{ are smooth a.e.}\}. \tag{199}
\]

Since the set \( \{x_\lambda\} \) is finite, considering the convolutions \( (u_\lambda, \nabla p_\lambda) \) with mollifiers it is proved that \( (u, \nabla p) \in \Phi^{-1}(\mathcal{Y}) \) and thus

\[
(u, \nabla p) = \Phi^{-1}((f, r_2, u_0)) \tag{200}
\]

(note that \( (u_\lambda|_{K_{\lambda}}, \nabla p_\lambda|_{K_{\lambda}}) \) is convergent as elements in \( \mathcal{X}_{K_{\lambda}} \)). Let \( (s, y) \in I \times (\mathbb{R}^3/\mathbb{Z}^3) \) be a singular point of \( (u, \nabla p) \). Take another finite decomposition \( K_y \cup (\bigcup_{\mu} K_{\lambda_{\mu}}) = \mathbb{R}^3/\mathbb{Z}^3 \) and construct \( (u_1, \nabla p_1) \) that is smooth at \( (s, y) \) and satisfies equation (7) a.e. Since

\[
(u, \nabla p) = \Phi^{-1}((f, r_2, u_0)) = (u_1, \nabla p_1) \tag{201}
\]

it follows that \( (u, \nabla p) \) is smooth at \( (s, y) \). This contradiction shows that \( (u, \nabla p) \in \mathcal{X} \). Since

\[
(f, r_2, u_0) = \Phi((u, \nabla p)) \in \mathcal{Y} \tag{202}
\]

it is concluded that \( \mathcal{X} \subset \mathcal{Y} \). Since the other inclusion is trivial the assertion follows. \( \square \)

**Remark 64.** Navier-Stokes conditions (see Remark 65) are used.

**Proof of Theorem 2.** Since \( (f, 0, u_0) \in \mathcal{X} = \mathcal{Y} \) (see Lemma 63) there exists \( p \in \mathcal{X} \) such that \( \Phi(p) = (f, 0, u_0) \), which defines a solution of (7). Further since \( \Phi : \mathcal{X} \to \mathcal{Y} \) is a homeomorphism the solution is unique. The assertion follows. \( \square \)

**Remark 65.** In the proof of Theorem 2 we used the following Navier-Stokes conditions (which are not assumptions) and conclude \( \Phi \) is a homeomorphism.

1. \( \Phi : \mathcal{X} \to \mathcal{Y} \) is a \( C^\infty \)-map such that \( d\Phi(p) : T_p \mathcal{X} \to T_{\Phi(p)} \mathcal{Y} \) is a linear isomorphism.

2. Each seminorm \( p_k \) of \( \mathcal{Y} \), which is induced from the relative topology from

\[
W^{k,k}_3 \times (W^{k,k+2}_3)_{\text{div}} \times H^{k+2}(\mathbb{R}^3/\mathbb{Z}^3)^3,
\]

is actually a norm, that is, it satisfies the condition that \( p_k(r) = 0 \) implies \( r = 0 \).

3. The above holds if we replace \( I \) with \( J \).
4. For any \( r \in \mathcal{I} \) there exists \( q \in \bigcup_p U_p^J \) such that \( \Phi^J(q)|_t = r(t) \) \( (t_0 \leq t < t_0 + t_1) \) for small \( t_1 \).

5. Let \((f, r_2, u_0) \in \mathcal{I}\). Let \( x \in \mathbb{R}^3/\mathbb{Z}^3 \). Then there exist a compact neighborhood \( K_x \) of \( x \) and \((u_{K_x}, \nabla p_{K_x}) \in \mathcal{I}_{K_x} \) satisfying equation (7) on \( I \times K_x \).

Proof of Corollary 3. By Theorem 2 there exists an unique solution \((u, \nabla p)\) of equation (35) on \([0, T]\) for any \( T > 0 \) and patching the solutions there exists an unique global solution of the equation on \( I \). The assertion follows. \( \square \)

8 Exact solutions

We obtain the exact solutions in the following way.

**Theorem 66.** Let \( \nu > 0 \). Let \( I := [0, T] \) \( (T > 0) \). Let \( u_0 \in \mathcal{C}^{\infty}(\mathbb{R}^3/\mathbb{Z}^3)^3 \) such that \( \nabla \cdot u_0 = 0 \) and let \( f \in W_3 \). Then the formal power series solution

\[
(u, \nabla p) = \left( \sum_m c_m t^m, \sum_m \nabla q_m t^m \right) \tag{204}
\]

of equation (7), where

\[
\begin{aligned}
c_m &= \sum_{L \in \mathbb{Z}^3} a_{L,m} e^{2\pi iL \cdot x} \quad (a_{L,m} \in \mathbb{C}^3), \\
q_m &= \sum_{L \in \mathbb{Z}^3} b_{L,m} e^{2\pi iL \cdot x} \quad (b_{L,m} \in \mathbb{C})
\end{aligned} \tag{205}
\]

are formal Fourier series, is unique.

**Proof.** From \( u|_{t=0} = u_0 \), it is obtained that \( c_0 = u_0 \). From

\[
\nabla \cdot (the \ first \ formula \ of \ equation \ (7)) \tag{206}
\]

and the second formula of equation (7), \( \Delta q_m \) (and thus \( \nabla q_m \)) is determined from \( c_0, \ldots, c_m \). Then from the first formula of equation (7), \( (m + 1)c_{m+1} \) is determined from \( \nabla q_m, c_0, \ldots, c_m \). By induction all \( c_m, \nabla q_m \) are determined from \( u_0 \). Thus the formal power series solution \((u, \nabla p)\) is unique. \( \square \)

**Remark 67.** In the above \( q_m \) is not unique.

Let \((u, \nabla p)\) be a smooth solution of (2). Expand \( u \) and \( p \) (not \( \nabla p \)) in the following way:

\[
(u, p) = \left( \sum_{m=0}^{\infty} \sum_{L \in \mathbb{Z}^3} \tilde{a}_{L,m} e^{2\pi iL \cdot x} e_m(t), \sum_{m=0}^{\infty} \sum_{L \in \mathbb{Z}^3} \tilde{b}_{L,m} e^{2\pi iL \cdot x} \tilde{e}_m(t) \right), \tag{207}
\]

where \( \tilde{a}_{L,m} \in \mathbb{C}^3 \), \( \tilde{b}_{L,m} \in \mathbb{C} \) and \( e_m(t) \) is the \( m \)-th orthogonal polynomial in \( L^2([0, T]) \). From this and Theorem 66 the smooth solution \((u, \nabla p)\) may be calculated.
9 Hodge Theory on $\mathbb{R}^3$

Let $k \in \mathbb{Z}_{\geq 0}$. Let $H^k(\mathbb{R}^3)$ be the Sobolev space. Let

$$H^\infty(\mathbb{R}^3) := \bigcap_k H^k(\mathbb{R}^3).$$

(208)

Let $\nabla \cdot (\cdot) : u \in H^\infty(\mathbb{R}^3)^3 \mapsto \nabla \cdot u \in H^\infty(\mathbb{R}^3)$. Let

$$(H^\infty(\mathbb{R}^3))^\nabla := \{ v \mid v = \nabla w \ (\exists w \in H^\infty(\mathbb{R}^3)) \}.$$  

(209)

The following is a special case of [4], Chapter VIII, Section 3, (3.2) Theorem.

**Theorem 68.** $H^\infty(\mathbb{R}^3)^3 = \text{Ker}(\nabla \cdot (\cdot)) \oplus (H^\infty(\mathbb{R}^3))^\nabla$.

The following is obtained from [9], Chapter VI, Section 7, Theorem (Sobolev’s Lemma).

**Theorem 69.** Let $K \subset \mathbb{R}^3$ be a compact set. Let $k,l \in \mathbb{Z}_{\geq 0}$ be such that $l > \frac{3}{2} + k$. Let $H^l(K)$ be the Sobolev space. Then there exists $C > 0$ such that

$$||u||_{C^k(K)} \leq C ||u||_{H^l(K)}.$$  

(210)

Let $k \in \mathbb{Z}_{\geq 0}$. Write as

$$\limesssup_{|x| \to \infty} |\partial_x^2 v(x)|^2$$

the supremum of $\limesssup_{j \to \infty} |\partial_x^2 v(x_j)|^2$, where $\{x_j\}$ runs all the sequences such that $|x_j| \to \infty$ as $j \to \infty$. Let

$$\Gamma^0_0(\mathbb{R}^3) := \{ v \in H^k(\mathbb{R}^3)^3 \mid ||v|| := ||v||^2_{H^k(\mathbb{R}^3)} + \sum_{|\alpha| \leq k} \limesssup_{|x| \to \infty} |\partial_x^\alpha v(x)|^2 < \infty \}.$$  

(212)

Let

$$\Gamma^k(\mathbb{R}^3) := \{ v \in \Gamma^0_0(\mathbb{R}^3) \mid \sum_{|\alpha| \leq k} \limesssup_{|x| \to \infty} |\partial_x^\alpha v(x)|^2 = 0 \},$$

(213)

and

$$\Gamma^\infty(\mathbb{R}^3) := \bigcap_k \Gamma^k(\mathbb{R}^3).$$  

(214)

**Lemma 70.** $H^\infty(\mathbb{R}^3) = \Gamma^\infty(\mathbb{R}^3)$.

**Proof.** Let $w \in H^\infty(\mathbb{R}^3)$. Let $K \subset \mathbb{R}^3$ be a compact neighbourhood of 0. Let $w^l(y) := w(x + y) \ (y \in K)$. By Theorem 69 it is obtained that for $l > \frac{3}{2} + k$ there exists $C > 0$ such that

$$||w^l||_{C^k(K)} \leq C ||w^l||_{H^l(K)},$$  

(216)
so that since \( w \in H^\infty(\mathbb{R}^3) \),
\[
||w(x + \cdot)||_{C^k(K)} \leq C||w(x + \cdot)||_{H^1(K)} \to 0 \ (|x| \to \infty).
\]
(217)

Thus
\[
\sum_{|\alpha| \leq k} \lim_{|x| \to \infty} \text{esssup} |\partial^\alpha_x w(x)|^2 = 0.
\]
(218)

It follows that \( w \in \Gamma^\infty(\mathbb{R}^3) \) and \( H^\infty(\mathbb{R}^3) \subset \Gamma^\infty(\mathbb{R}^3) \). Since the other inclusion is trivial the assertion follows.

Let
\[
(\Gamma^\infty(\mathbb{R}^3)_\nu := \{ v \mid v = \nabla w \ (\exists w \in \Gamma^\infty(\mathbb{R}^3)) \}).
\]
(219)

**Corollary 71.** \( \Gamma^\infty(\mathbb{R}^3)^3 = (\text{Ker}(\nabla \cdot (\cdot)) \cap \Gamma^\infty(\mathbb{R}^3)^3) \oplus (\Gamma^\infty(\mathbb{R}^3))_\nu \).

**Proof.** By Lemma 70 \( H^\infty(\mathbb{R}^3) = \Gamma^\infty(\mathbb{R}^3) \) so that \( \Gamma^\infty(\mathbb{R}^3)^3 = H^\infty(\mathbb{R}^3)^3 \) and \( (\text{Ker}(\nabla \cdot (\cdot)) \cap \Gamma^\infty(\mathbb{R}^3)^3) \oplus (\Gamma^\infty(\mathbb{R}^3))_\nu = \text{Ker}(\nabla \cdot (\cdot)) \oplus (H^\infty(\mathbb{R}^3))_\nu \). By Theorem 68 the assertion follows.

**10 Linear evolution equations on \( \mathbb{R}^3 \)**

Let \( \mathcal{P} : \Gamma^\infty(\mathbb{R}^3) \to \text{Ker}(\nabla \cdot (\cdot)) \) be the projection. For \( n \in \mathbb{N} \) let
\[
W_n := C^\infty(I, \Gamma^\infty(\mathbb{R}^3)^n).
\]
(220)

We shall prove the existence and uniqueness of a solution of the equation
\[
\begin{aligned}
\dot{h} + \mathcal{P}(u \cdot \nabla)\mathcal{P}h + (\mathcal{P}h \cdot \nabla)u - \nu \Delta \mathcal{P}h - g = 0, \\
h(0) \in \Gamma^\infty(\mathbb{R}^3)^3,
\end{aligned}
\]
(221)

for \( g \in C^\infty(I, \Gamma^\infty(\mathbb{R}^3)^3) \).

Let \( k \in \mathbb{Z}_{\geq 0} \). Let \( I := [0, T] \) and \( u \in C^\infty(I, \Gamma^\infty(\mathbb{R}^3)^3) \). Define a linear operator \( A_k(t) \) on \( \Gamma^k(\mathbb{R}^3)^3 \) by
\[
-A_k(t)h := \mathcal{P}(u \cdot \nabla)\mathcal{P}h + (\mathcal{P}h \cdot \nabla)u - \nu \Delta \mathcal{P}h
\]
(222)

for \( h \in \Gamma^\infty(\mathbb{R}^3)^3 \). Since the adjoint \( (A_k(t))^* \) of \( A_k(t) \) is densely defined \( A_k(t) \) is closable. Let \( A_k(t) \) be the closure of \( A_k(t) \). For \( k_1, k_2 \in \mathbb{Z}_{\geq 0} \cup \{ \infty \} \) let
\[
W_{n}^{k_1,k_2} := C^{k_1}(I, \Gamma^{k_2}(\mathbb{R}^3)^n).
\]
(223)

**Lemma 72.** Let \( I := [0, T] \ (T > 0) \). Let \( \nu > 0 \). Assume \( u \in W_3 \). Then the equation
\[
\dot{h} - A_k(t)h = 0
\]
(224)

has at most one solution \( h \in W_{3}^{1,k} \) for any initial condition \( h(0) \in \Gamma^\infty(\mathbb{R}^3)^3 \).
Proof. Let $h \in \Gamma^\infty(\mathbb{R}^3)^3$. Observe that $-A_k(t)h = \mathcal{P}(-\nu(\nabla - A(t))^2)\mathcal{P}h + \mathcal{P}\mathbb{A}(t)\mathcal{P}h$ for some $W_1$-coefficiential $3 \times 1$ and $3 \times 3$ matrices $A(t), \mathbb{A}(t)$. Thus by an elementary argument

$$
\text{Re} < -A_k(t)h, h >_{\Gamma^k(\mathbb{R}^3)^3} \geq -c < h, h >_{\Gamma^k(\mathbb{R}^3)^3} \tag{225}
$$

for some $c > 0$. Taking limit formula (225) also holds for $h \in D(-A_k(t))$, where $D(-A_k(t))$ is the domain of $-A_k(t)$.

Let $h$ be the solution of equation (224) and $H = e^{-ct}h$. Observe that $h$ depends on $t$. Then

$$
< \dot{H}, H >_{\Gamma^k(\mathbb{R}^3)^3} = < (A_k(t) - c)H, H >_{\Gamma^k(\mathbb{R}^3)^3} . \tag{226}
$$

Assume $h(0) = 0$. Let $t_0 \in [0, T]$. By formula (225)

$$
||H(t_0)||^2_{\Gamma^k(\mathbb{R}^3)^3} \leq ||H(0)||^2_{\Gamma^k(\mathbb{R}^3)^3} \tag{227}
$$

$$
= 2 \int_0^{t_0} \text{Re} < (A_k(t) - c)H, H >_{\Gamma^k(\mathbb{R}^3)^3} \, dt \tag{228}
$$

$$
\leq 0. \tag{229}
$$

It follows that

$$
||H(t_0)||^2_{\Gamma^k(\mathbb{R}^3)^3} \leq ||H(0)||^2_{\Gamma^k(\mathbb{R}^3)^3} = 0. \tag{230}
$$

Hence $h(t_0) = 0$. Since $t_0 \in [0, T]$ is arbitrary $h = 0$. Assume $h_1, h_2$ are two solutions. Then $h_2(0) - h_1(0) = 0$ and $h_2 - h_1$ is a solution of the equation (224). Thus the above argument shows that $h_2 - h_1 = 0$. From this the assertion follows.

\begin{lemma}
Lemma 73. $A_k(t)$ generates a $C^0$-semigroup for each $t \in I$.
\end{lemma}

Proof. Take $c$ as in Lemma 72. Since by definition the domain of $A_k(t)$ is dense, so is that of $(A_k(t) - c)$. Observe that

$$
\text{Re} < (A_k(t) - c)h, h >_{\Gamma^k(\mathbb{R}^3)^3} \leq 0 \tag{231}
$$

for all $h \in D(A_k(t) - c)$, where $D(A_k(t) - c)$ is the domain of $(A_k(t) - c)$, so $(A_k(t) - c)$ is dissipative. In particular the image of $\text{Id} - (A_k(t) - c)$ is closed. The adjoint operator $(\text{Id} - (A_k(t) - c))^*$ of $\text{Id} - (A_k(t) - c)$ is clearly injective. Hence the image of $\text{Id} - (A_k(t) - c)$ is the whole space $\Gamma^k(\mathbb{R}^3)^3$. By Theorem 18, $(A_k(t) - c)$ generates a contraction $C^0$-semigroup. The assertion follows.

\begin{lemma}
Lemma 74. Let $k, k' \in \mathbb{Z}_{\geq 0}$. Let \( \{ e^{sA_k(t)} \}_{s \geq 0} \) be a $C^0$-semigroup on $\Gamma^k(\mathbb{R}^3)^3$ generated by $A_k(t)$ for each $t$. Then \( e^{sA_k(t)}h_0 = e^{sA_{k'}(t)}h_0 \) for any $h_0 \in \Gamma^\infty(\mathbb{R}^3)^3$ and $s \in I := [0, T]$.
\end{lemma}
Proof. Assume without loss of generality \( k \leq k' \). Observe that by assumption \( h_0 \in \Gamma^\infty(\mathbb{R}^3)^3 \) so that \( h_1(s, x) := e^{sA_k(t)}h_0 \in W^{1,k}_{3} \) and \( h_2(s, x) := e^{sA_{k'}(t)}h_0 \in W^{1,k'}_{3} \). Then since \(-A_k(t)|_{\Gamma^k(\mathbb{R}^3)^3} = -A_{k'}(t)\) it follows that \( h_1 \) and \( h_2 \) are solutions of

\[
\begin{align*}
\frac{\partial h}{\partial s} + A_k(t)h &= 0, \\
h(0) &= h_0. 
\end{align*}
\] (232)

Since \( A_k(t) \) is dissipative an elementary argument shows that the solution \( h \in W^{1,k} \) of this equation is unique. Thus \( e^{sA_k(t)}h_0 = h_1(s, x) = h_2(s, x) = e^{sA_{k'}(t)}h_0 \) for \( s \in I \). The assertion follows. \( \square \)

Let \( B(\Gamma^k(\mathbb{R}^3)^3) \) be the set of continuous linear operators on \( \Gamma^k(\mathbb{R}^3)^3 \).

**Theorem 75.** Assume

\[
\| \prod_{j=1}^{I}(\lambda - A_k(t_j))^{-1} \|_{B(\Gamma^k(\mathbb{R}^3)^3)} \leq M_k(\lambda - \beta_k)^j (\lambda > \beta_k). \] (233)

for \( 0 \leq t_1 \leq \cdots \leq t_j \leq T \) \((J = 1, 2, \ldots)\), where the product \( \prod \) is time-ordered, i.e. a factor with larger \( t_j \) stands to the left of ones with smaller \( t_j \). Then there exists an operator-valued function \( U(t,s) \) \((0 \leq s \leq t \leq T)\) on \( \Gamma^\infty(\mathbb{R}^3)^3 \) that satisfies the following.

(a) \((s, t) \mapsto U(t,s)h_0 \) \((h_0 \in \Gamma^\infty(\mathbb{R}^3)^3)\) is continuous, \(U(s,s) = \text{Id}, \)

\[
\| U(t,s) \|_{B(\Gamma^k(\mathbb{R}^3)^3)} \leq M_k e^{\beta_k(t-s)}. \] (234)

(b) If \( s \leq r \leq t \) then \( U(t,s) = U(t,r)U(r,s). \)

(c) For \( h_0 \in \Gamma^\infty(\mathbb{R}^3)^3 \) and \( s \in [0, T), \)

\[
D^+ U(t,s)h_0|_{t=s} = A_0(s)h_0, \] (235)

where \( D^+ \) denotes the right derivative.

(d) For \( h_0 \in \Gamma^\infty(\mathbb{R}^3)^3 \) and \( 0 \leq s \leq t \leq T, \)

\[
\frac{\partial}{\partial s} U(t,s)h_0 = -U(t,s)A_0(s)h_0. \] (236)

(e) For \( h_0 \in \Gamma^\infty(\mathbb{R}^3)^3 \) and \( 0 \leq s \leq t \leq T, \)

\[
\frac{\partial}{\partial t} U(t,s)h_0 = A_0(t)U(t,s)h_0. \] (237)

Proof. By Lemma 73, \( A_k(t) \) generates a \( C^0 \)-semigroup \( \{e^{sA_k(t)}\}_{s \geq 0} \) and since \( \Gamma^\infty(\mathbb{R}^3)^3 \) is dense in \( \Gamma^k(\mathbb{R}^3)^3 \), it is obtained by Lemma 74 that

\[
e^{sA_k(t)}|_{\Gamma^k(\mathbb{R}^3)^3} = e^{sA_k(t)}. \] (238)
By Lemma 8,

\[ \left\| \prod_{j=1}^{J} e^{s_j A_0(t_j)} \right\|_{\Gamma^k(\mathbb{R}^3)^3} \right\|_{B(\Gamma^k(\mathbb{R}^3)^3)} \]

\[ \| \prod_{j=1}^{J} e^{s_j A_k(t_j)} \|_{B(\Gamma^k(\mathbb{R}^3)^3)} \]  

(239)

\[ \| \prod_{j=1}^{J} e^{s_j A_k(t_j)} \|_{B(\Gamma^k(\mathbb{R}^3)^3)} \]

(240)

\[ \beta_k \left( \sum_{j=1}^{J} s_j \right) \leq M_k e \]  

(241)

for \( 0 \leq t_1 \leq \cdots \leq t_J \leq T \) and \( s_j \geq 0 \). The product \( \prod \) is time-ordered, i.e. a factor with larger \( t_j \) stands to the left of ones with smaller \( t_j \).

Observe that \( A_0(t)|_{\Gamma^k(\mathbb{R}^3)^3} = A_k(t) \). Let \( L(\Gamma^{k+2}(\mathbb{R}^3)^3, \Gamma^k(\mathbb{R}^3)^3) \) be the set of continuous linear operators from \( \Gamma^{k+2}(\mathbb{R}^3)^3 \) to \( \Gamma^k(\mathbb{R}^3)^3 \). Let \( A_{0,n}(t) = A_0(T|nt/T|/n) \), where \([\cdot]\) denotes the Gauss symbol. Then

\[ \| A_{0,n}(t) - A_0(t) \|_{L(\Gamma^{k+2}(\mathbb{R}^3)^3, \Gamma^k(\mathbb{R}^3)^3)} \rightarrow 0 \ (n \rightarrow \infty) \]  

(242)

Let

\[ U_n(t, s) = e^{(t-s)A_0(k'T/n)}, \ (k'T/n \leq s \leq (k'+1)T/n), \]  

(243)

\[ U_n(t, s) = e^{(t-l'T/n)A_0(l'T/n)} e^{(T/n)A_0((l'-1)T/n)} \cdots e^{(T/n)A_0((k'+1)T/n)} e^{((k'+1)T/n - s)A_0(k'T/n)}, \]  

(244)

\[ (k'T/n \leq s < (k'+1)T/n, l'T/n \leq t < (l'+1)T/n, k' < l'). \]  

(245)

By inequality (239)-(241),

\[ \| U_n(t, s) \|_{B(\Gamma^k(\mathbb{R}^3)^3)} \leq M_k e^{\beta_k(t-s)}. \]  

(247)

Thus \( U_n(t, s) \) satisfies (a), (b). For \( h_0 \in \Gamma^{\infty}(\mathbb{R}^3)^3 \) and \( t \neq k''T/n \) (\( k'' = 0, 1, \ldots, n \)),

\[ \frac{\partial}{\partial t} U_n(t, s) h_0 = A_{0,n}(t) U_n(t, s) h_0, \]  

(248)

and for \( h_0 \in \Gamma^{\infty}(\mathbb{R}^3)^3 \) and \( s \neq l''T/n \) (\( l'' = 0, 1, \ldots, n \)),

\[ \frac{\partial}{\partial s} U_n(t, s) h_0 = -U_n(t, s) A_{0,n}(s) h_0. \]  

(249)
Let \( h_0 \in \Gamma^\infty(\mathbb{R}^3)^3 \). Then by inequality (247),

\[
\|U_n(t, s)h_0 - U_m(t, s)h_0\|_{\Gamma^k(\mathbb{R}^3)^3}
\]

(250)

\[
= \left\| - \int_s^t \frac{\partial}{\partial r} U_n(t, r)U_m(r, s)h_0 dr \right\|_{\Gamma^k(\mathbb{R}^3)^3}
\]

(251)

\[
= \left\| \int_s^t U_n(t, r)(A_{0,n}(r) - A_{0,m}(r))U_m(r, s)h_0 dr \right\|_{\Gamma^k(\mathbb{R}^3)^3}
\]

(252)

\[
\leq M_k M_{k+2} e^{\max(\beta_k, \beta_{k+2})(t-s)} \|h_0\|_{\Gamma^{k+2}(\mathbb{R}^3)^3}
\]

(253)

\[
\times \int_s^t \|A_{0,n}(r) - A_{0,m}(r)\|_{L(\Gamma^{k+2}(\mathbb{R}^3)^3, \Gamma^k(\mathbb{R}^3)^3)} dr
\]

(254)

\[
\to 0 \quad (n, m \to \infty).
\]

(255)

\( k \in \mathbb{Z}_{\geq 0} \) is arbitrary and thus for any \( h_0 \in \Gamma^\infty(\mathbb{R}^3)^3 \) and \( 0 \leq s \leq t \leq T \),

\[
U(t, s)h_0 := \lim_{n \to \infty} U_n(t, s)h_0
\]

(256)

exists. Note that \((s, t) \mapsto U(t, s)\) is continuous because the convergence is uniform in \( 0 \leq s \leq t \leq T \). Since \( U_n(t, s) \) satisfies (a), (b) so does \( U(t, s) \). For \( h_0 \in \Gamma^\infty(\mathbb{R}^3)^3 \),

\[
\|U_n(t, s)h_0 - e^{(t-s)A_0}h_0\|_{\Gamma^k(\mathbb{R}^3)^3}
\]

(257)

\[
= \left\| - \int_s^t \frac{\partial}{\partial r} U_n(t, r)e^{(r-s)A_0}h_0 dr \right\|_{\Gamma^k(\mathbb{R}^3)^3}
\]

(258)

\[
= \left\| \int_s^t U_n(t, r)(A_{0,n}(r) - A_0(r))e^{(r-s)A_0}h_0 dr \right\|_{\Gamma^k(\mathbb{R}^3)^3}
\]

(259)

\[
\leq M_k M_{k+2} e^{\max(\beta_k, \beta_{k+2})(t-s)} \|h_0\|_{\Gamma^{k+2}(\mathbb{R}^3)^3}
\]

(260)

\[
\times \int_s^t \|A_{0,n}(r) - A_0(r)\|_{L(\Gamma^{k+2}(\mathbb{R}^3)^3, \Gamma^k(\mathbb{R}^3)^3)} dr.
\]

(261)

It follows that for \( h_0 \in \Gamma^\infty(\mathbb{R}^3)^3 \),

\[
\|U(t, s)h_0 - e^{(t-s)A_0}h_0\|_{\Gamma^k(\mathbb{R}^3)^3}
\]

(262)

\[
\leq M_k M_{k+2} e^{\max(\beta_k, \beta_{k+2})(t-s)} \|h_0\|_{\Gamma^{k+2}(\mathbb{R}^3)^3}
\]

(263)

\[
\times \int_s^t \|A_{0,n}(r) - A_0(r)\|_{L(\Gamma^{k+2}(\mathbb{R}^3)^3, \Gamma^k(\mathbb{R}^3)^3)} dr.
\]

(264)

Since \( k \in \mathbb{Z}_{\geq 0} \) is arbitrary this proves (c). Similarly it is shown that for \( h_0 \in \Gamma^\infty(\mathbb{R}^3)^3 \),

\[
D^- U(t, s)h_0|_{s=t} = -A_0(t)h_0,
\]

(265)

where \( D^- \) denotes the left derivative. For \( h_0 \in \Gamma^\infty(\mathbb{R}^3)^3 \), \( s < t \) it is obtained
by (c) and the continuity of $U(t, s)$ that

$$
\frac{1}{\epsilon}(U(t, s + \epsilon)h_0 - U(t, s)h_0) = U(t, s + \epsilon) - U(t, s)
$$

$$
\rightarrow -U(t, s)A_0(s)h_0 (\epsilon \rightarrow +0).
$$

For $h_0 \in \Gamma^\infty(\mathbb{R}^3)^3$, $s \leq t$ it is obtained by formula (265) that

$$
\frac{1}{\epsilon}(U(t, s)h_0 - U(t, s - \epsilon)h_0) = U(t, s) - U(t, s - \epsilon)
$$

$$
\rightarrow -U(t, s)A_0(s)h_0 (\epsilon \rightarrow +0).
$$

They prove (d). (e) follows from (c) and (d).

\[ \square \]

**Lemma 76.** Let $I := [0, T]$ ($T > 0$). Let $\nu > 0$. Assume $u, g \in W_3$. Then the equation

$$
\hat{h} - A_0(t)h - g = 0
$$

has a solution $h \in W_3^{1, \infty}$ that is uniquely determined from $h(0) \in \Gamma^\infty(\mathbb{R}^3)^3$.

**Proof.** By Theorem 75 there exists $U(t, s)$ satisfying (a)-(e). Let

$$
h := U(t, 0)h(0) + \int_0^t U(t, s)g(s)ds.
$$

Then $h \in W_3^{1, \infty}$ and it is a solution of equation (272). Let $h_1, h_2$ be two solutions of equation (272) then $h_2 - h_1$ is a solution of

$$
\begin{cases}
\hat{h} - A_0(t)h = 0, \\
|_{t=0} = 0.
\end{cases}
$$

By Lemma 72 this equation has the unique solution 0. Thus $h_2 - h_1 = 0$ and equation (272) has a solution $h \in W_3^{1, \infty}$ that is uniquely determined from $h(0) \in \Gamma^\infty(\mathbb{R}^3)^3$. The assertion follows.

\[ \square \]

**Theorem 77.** Let $I := [0, T]$ ($T > 0$). Let $\nu > 0$. Assume $u, g \in W_3$. Then the equation

$$
\hat{h} + \mathcal{P}((u \cdot \nabla)\mathcal{P}h + (\mathcal{P}h \cdot \nabla)u - \nu\Delta \mathcal{P}h) - g = 0
$$

has a solution $h \in W_3$ that is uniquely determined from $h(0) \in \Gamma^\infty(\mathbb{R}^3)^3$.

**Proof.** By Lemma 76, equation (275) has an unique solution $h$ in $W_3^{1, \infty}$. Since $h$ satisfies equation (275), if $h \in W_3^{l, \infty}$ for $l \in \mathbb{N}$, it follows that $\hat{h} \in W_3^{l-1, \infty}$, that is, $h \in W_3^{l+1, \infty}$. Hence by induction it is proved that $h \in W_3$. The assertion follows.

\[ \square \]
11 Navier-Stokes equations on $\mathbb{R}^3$

Let $\nu > 0$ and let $I := [0, T]$ ($T > 0$). Let

$$(W_1)_\nabla := \{ v \mid v = \nabla w \ (\exists w \in W_1) \}. \tag{276}$$

We prove the following theorem.

**Theorem 78.** Let $\nu > 0$. Let $I := [0, T]$ ($T > 0$). Let $u_0 \in \Gamma^\infty(\mathbb{R}^3)^3$ such that $\nabla \cdot u_0 = 0$ and let $f \in W_3$. Then there exists a unique $(u, \nabla p) \in W_3 \times (W_1)_\nabla$ such that

$$\begin{cases}
\frac{\partial u}{\partial t} = -(u \cdot \nabla)u + \nu \Delta u - \nabla p + f, \\
\nabla \cdot u = 0, \\
u |_{t=0} = u_0, \\
limesssup_{|x| \to \infty} |\partial_2^2 u(t, x)| = 0 \ (\forall t, \alpha). \tag{277}
\end{cases}$$

The proof of Theorem 78 is essentially the same as that of Theorem 2. We describe the details.

Let

$$\Phi : W_3 \times (W_1)_\nabla \to W_3 \times (W_3)_{\text{div}} \times \Gamma^\infty(\mathbb{R}^3)^3 \tag{278}$$

be given by

$$(u, P) \mapsto \begin{bmatrix}
\dot{u} + (u \cdot \nabla)u - \nu \Delta u + P \\
\nabla \cdot u \\
u(0)
\end{bmatrix}, \tag{279}$$

where

$$(W_3)_{\text{div}} := \{ v \mid v = \nabla \cdot w \ (\exists w \in W_3) \}. \tag{280}$$

We introduce to $(W_1)_\nabla$ the relative topology as a subset of $W_3$ and to $(W_3)_{\text{div}}$ the relative topology as a subset of $W_1$. Then $\Phi$ is $C^\infty$ in $(u, P)$ in the sense of Fréchet derivatives (see Definition 31). We begin with proving the following lemma:

**Lemma 79.** The map $d\Phi((u, P)) : (h, \beta) \mapsto \begin{bmatrix}
\dot{h} + (u \cdot \nabla)h + (h \cdot \nabla)u - \nu \Delta h + \beta \\
\nabla \cdot h \\
h(0)
\end{bmatrix}$ is a linear isomorphism from $W_3 \times (W_1)_\nabla$ to $\{(a, b, c) \in W_3 \times (W_3)_{\text{div}} \times \Gamma^\infty(\mathbb{R}^3)^3 \mid \nabla \cdot c = b(0)\}$. \tag{281}
Proof. By calculation \( d\Phi \) is given by the map (281). Let
\[
(a, b, c) \in \{(a, b, c) \in W_3 \times (W_3)_{\text{div}} \times \Gamma^\infty(\mathbb{R}^3)^3 \mid \nabla \cdot c = b(0)\}. \tag{284}
\]

Let
\[
\dot{h} + (u \cdot \nabla)h + (h \cdot \nabla)u - \nu \Delta h + \beta = a, \tag{285}
\]
\[
\nabla \cdot h = b, \tag{286}
\]
\[
h(0) = c. \tag{287}
\]

From equation (286), \( h \in W_3 \) is determined up to \( \text{Ker}(\nabla \cdot \cdot) \). Since \( \beta \in (W_1)_\nabla \), by Corollary 71, Theorem 77 and an elementary argument \( h \in W_3 \) is uniquely determined from equation (285) and equation (287). Then from equation (285), \( \beta \in (W_1)_\nabla \) is uniquely determined. From this the assertion follows. \( \square \)

Let
\[
\mathcal{X} := W_3 \times (W_1)_\nabla \tag{288}
\]

and
\[
\mathcal{Y} := \{r := (f, r_2, u_0) \in W_3 \times (W_3)_{\text{div}} \times \Gamma^\infty(\mathbb{R}^3)^3 \mid \nabla \cdot u_0 = r_2(0)\}. \tag{289}
\]

Then \( \Phi \) is a map from \( \mathcal{X} \) to \( \mathcal{Y} \). Let
\[
\mathcal{Y} := \Phi(\mathcal{X}). \tag{290}
\]

Let \( p = (u, P) \in \mathcal{X} \) be an arbitrary point.

**Lemma 80.** \( d\Phi(p) \) is a topological isomorphism.

**Proof.** Note that \( \Phi \) is a \( C^1 \)-map from \( \mathcal{X} \) to \( \mathcal{Y} \) and from Lemma 79
\[
d\Phi(p) : (h, \beta) \mapsto \begin{bmatrix} \dot{h} + (u \cdot \nabla)h + (h \cdot \nabla)u - \nu \Delta h + \beta \\ \nabla \cdot h \\ h(0) \end{bmatrix} \tag{291}
\]
is a linear isomorphism and by the open mapping principle a topological linear isomorphism from
\[
T_p \mathcal{X} = \{(h, \beta) \in W_3 \times (W_1)_\nabla \} \tag{292}
\]
to
\[
T_{\Phi(p)} \mathcal{Y} = \{(a, b, c) \in W_3 \times (W_3)_{\text{div}} \times \Gamma^\infty(\mathbb{R}^3)^3 \mid \nabla \cdot c = b(0)\}. \tag{293}
\]
The assertion follows. \( \square \)

**Lemma 81.** \( \mathcal{Y} \) is a Fréchet manifold.

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Proof. By Lemma 80 \( d\Phi(p) \) is a topological isomorphism so that, since \( p \) is arbitrary, by Lemma 30 \( \Phi : \mathcal{X} \to \mathcal{Y} \) is an open map. Since \( \mathcal{Y} = \Phi(\mathcal{X}) \) it is a Fréchet manifold. The assertion follows.

Let

\[
\varphi_p : \mathcal{X} \xrightarrow{\sim} T_p \mathcal{X}
\]

and

\[
\psi_{\Phi(p)} : \mathcal{Y} \xrightarrow{\sim} T_{\Phi(p)} \mathcal{Y}
\]

be canonical isomorphisms. Let

\[
\mathcal{U}_p := \varphi_p^{-1}(T_p \mathcal{X}) (= \mathcal{X})
\]

and

\[
\mathcal{V}_{\Phi(p)} := \psi_{\Phi(p)}^{-1}(T_{\Phi(p)} \mathcal{Y}).
\]

Observe that by Lemma 80

\[
d\Phi(p) : T_p \mathcal{X} \xrightarrow{\sim} T_{\Phi(p)} \mathcal{Y}
\]

is a topological linear isomorphism. Let \( k \in \mathbb{N} \). Let

\[
(W^k_{3, k+2})_{\text{div}} := \{ v \mid v = \nabla \cdot w \ (\exists w \in W^k_{3, k+2}) \}.
\]

Then introduce to \((W^k_{3, k+2})_{\text{div}}\) the relative topology as a subset of \( W^{k, k+1}_1 \) and to \( \mathcal{Y} \) the relative topology as a subset of

\[
W^k_3 \times (W^k_{3, k+2})_{\text{div}} \times \Gamma^{k+2}(\mathbb{R}^3)^3,
\]

which defines a seminorm \( p_k \) (it is in fact a norm) of \( \mathcal{Y} \). Let the system of open sets of the topology of \( \mathcal{Y} \) be

\[
\{ \mathcal{O} \}.
\]

Introduce to \( T_{\Phi(p)} \mathcal{Y} \) the topology induced from \( \theta_{\Phi(p)} := (\psi_{\Phi(p)})^{-1}|_{T_{\Phi(p)} \mathcal{Y}} \), i.e.

\[
\{ \theta_{\Phi(p)}^{-1}(\mathcal{O}) \},
\]

to \( T_p \mathcal{X} \) the topology induced from \( \theta_{\Phi(p)} \circ d\Phi(p) \), i.e.

\[
\{ (d\Phi(p))^{-1} \circ \theta_{\Phi(p)}^{-1}(\mathcal{O}) \},
\]

to \( \mathcal{U}_p \) the topology induced from \( \theta_{\Phi(p)} \circ d\Phi(p) \circ \varphi_p|_{\mathcal{U}_p} \), i.e.

\[
\{ (\varphi_p|_{\mathcal{U}_p})^{-1} \circ (d\Phi(p))^{-1} \circ \theta_{\Phi(p)}^{-1}(\mathcal{O}) \},
\]

and to \( \mathcal{V}_{\Phi(p)} \) the topology induced from \( \theta_{\Phi(p)} \circ \psi_{\Phi(p)}|_{\mathcal{V}_{\Phi(p)}} \), i.e.

\[
\{ (\psi_{\Phi(p)}|_{\mathcal{V}_{\Phi(p)}})^{-1} \circ \theta_{\Phi(p)}^{-1}(\mathcal{O}) \}.
\]
Lemma 82. $U_p = \mathcal{X}$ and $V_{\Phi(p)} = \mathcal{Y}$.

Proof. From the definitions, using the canonical isomorphisms (294) and (295), the assertion follows. \qed

Remark 83. $\Phi$ satisfies Navier-Stokes condition 1. That $p_k$ is a norm is Navier-Stokes condition 2. (See Remark 108.)

By Lemma 82 $U_p = \mathcal{X}$ and $V_{\Phi(p)} = \mathcal{Y}$. The above topologies are induced by a seminorm $p_k$ such that $p_k(r)$ implies $r = 0$ (for $r \in \mathcal{Y}$). Also $U_p$ and $V_{\Phi(p)}$ are equipped with the ordinary topologies as $\mathcal{X}$ and $\mathcal{Y}$. Then $q \rightarrow d\Phi(q)$ is continuous with respect to the ordinary topology of $U_p$ and the topology of $T_{\Phi(p)}V_{\Phi(p)} (= T_{\Phi(p)}\mathcal{Y})$ induced by $p_k$.

Lemma 84. Let $\mathcal{X}_p$ be a sufficiently small convex neighbourhood of $p \in \mathcal{X}$ identified with the corresponding neighbourhood of $0 \in T_q\mathcal{X}_p$ ($q \in \mathcal{X}_p$). Let $\{q_j\} \subset \mathcal{X}_p$ be a sequence. Then there exists a subsequence of $\{q_j\}$ convergent to $0 \in \mathcal{X}_p$ with respect to the ordinary topology.

Proof. Let $k_1, k_2 \in \mathbb{Z}_{\geq 0}$. Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^3$ and let

$$
\Gamma^{k_2}(\Omega) := \{h|\Omega \mid h \in \Gamma^{k_2}(\mathbb{R}^3)\},
$$

$$
W^{k_1,k_2}_{\Omega,n} := C^{k_1}(I, \Gamma^{k_2}(\Omega)^n),
$$

$$
(W^{k_1,k_2}_{\Omega,1})^\nabla := \{v \mid v = \nabla w \ (\exists w \in W^{k_1,k_2+1}_{\Omega,1})\}.
$$

Let $(u_j, P_j) := q_j$. Let $K \subset \mathbb{R}^3$ be a compact set. By Theorem 41 possibly passing to a subsequence $(\partial_t^{k_1}u_j(t)|_K, \partial_t^{k_1}P_j(t)|_K)$ is convergent for fixed $t \in I$. Let

$$
(\partial_t^{k_1}u(t)|_K, \partial_t^{k_1}P(t)|_K) := \lim_{j \rightarrow \infty} (\partial_t^{k_1}u_j(t)|_K, \partial_t^{k_1}P_j(t)|_K).
$$

By diagonal argument the same formula holds for any $t_0 \in I'$, where $I'$ is a dense countable subset of $I$. By assumption $\|((\partial_t^{k_1+1}\partial_x^{\alpha}u_j, \partial_t^{k_1+1}\partial_x^{\alpha}P_j))\|_{W^{0,0}_{k,3}\times(W^{0,1}_{k,1})^\nabla}$ ($|\alpha| \leq k_2$) is bounded on $I$ so that $(\partial_t^{k_1}\partial_t^{\alpha}u(t_0), \partial_t^{k_1}\partial_t^{\alpha}P(t_0))$ ($t_0 \in I'$) extends to a continuous function

$$
(\lim_{t_0 \rightarrow t} \partial_t^{k_1}\partial_t^{\alpha}u(t_0), \lim_{t_0 \rightarrow t} \partial_t^{k_1}\partial_t^{\alpha}P(t_0)),
$$

where $t \in I$. The resulting function is denoted by the same symbol. Since $\|((\partial_t^{k_1+1}\partial_x^{\alpha}u_j, \partial_t^{k_1+1}\partial_x^{\alpha}P_j))\|_{W^{0,0}_{k,3}\times(W^{0,1}_{k,1})^\nabla}$ is bounded there exists $M > 0$ such that

$$
\|((\partial_t^{k_1}u_j(t)|_K, \partial_t^{k_1}P_j(t)|_K) - (\partial_t^{k_1}u_j(t_0)|_K, \partial_t^{k_1}P_j(t_0)|_K))\|_{W^{0,k_2}_{k,3}\times(W^{0,k_2+1}_{k,1})^\nabla}
$$

$$
\leq M|t - t_0|
$$

(312)
\[
\|(\partial_t^{k_1} u(t) |_{\mathbb{K}}, \partial_t^{k_1} P(t) |_{\mathbb{K}}) - (\partial_t^{k_1} u(t_0) |_{\mathbb{K}}, \partial_t^{k_1} P(t_0) |_{\mathbb{K}})\|_{W^{0,k_2}_{\mathbb{K},3} \times (W^{0,k_2+1}_{\mathbb{K},1})} \leq M |t - t_0|.
\] (313)

For any \( t \in I \) and any \( \epsilon > 0 \) there exists \( t_0 \in A \) such that \( M |t - t_0| < \epsilon \) so that by above for large \( j \)
\[
\|(\partial_t^{k_1} u_j(t) |_{\mathbb{K}}, \partial_t^{k_1} P_j(t) |_{\mathbb{K}}) - (\partial_t^{k_1} u(t) |_{\mathbb{K}}, \partial_t^{k_1} P(t) |_{\mathbb{K}})\|_{W^{0,k_2}_{\mathbb{K},3} \times (W^{0,k_2+1}_{\mathbb{K},1})} \leq \|(\partial_t^{k_1} u_j(t) |_{\mathbb{K}}, \partial_t^{k_1} P_j(t) |_{\mathbb{K}}) - (\partial_t^{k_1} u_j(t_0) |_{\mathbb{K}}, \partial_t^{k_1} P_j(t_0) |_{\mathbb{K}})\|_{W^{0,k_2}_{\mathbb{K},3} \times (W^{0,k_2+1}_{\mathbb{K},1})} + \|(\partial_t^{k_1} u_j(t_0) |_{\mathbb{K}}, \partial_t^{k_1} P_j(t_0) |_{\mathbb{K}}) - (\partial_t^{k_1} u(t_0) |_{\mathbb{K}}, \partial_t^{k_1} P(t_0) |_{\mathbb{K}})\|_{W^{0,k_2}_{\mathbb{K},3} \times (W^{0,k_2+1}_{\mathbb{K},1})} + \|(\partial_t^{k_1} u(t_0) |_{\mathbb{K}}, \partial_t^{k_1} P(t_0) |_{\mathbb{K}}) - (\partial_t^{k_1} u(t) |_{\mathbb{K}}, \partial_t^{k_1} P(t) |_{\mathbb{K}})\|_{W^{0,k_2}_{\mathbb{K},3} \times (W^{0,k_2+1}_{\mathbb{K},1})} \leq \epsilon.
\] (314)

Thus possibly passing to a subsequence for \( t \in I \)
\[
(\partial_t^{k_1} u_j(t) |_{\mathbb{K}}, \partial_t^{k_1} P_j(t) |_{\mathbb{K}}) \to (\partial_t^{k_1} u(t) |_{\mathbb{K}}, \partial_t^{k_1} P(t) |_{\mathbb{K}}) \quad (j \to \infty).
\] (320)

Since \( \mathbb{K} \) is compact the above formula also shows that the convergence is uniform. Observe that \( k_1, k_2 \) are arbitrary.

Let \( n \in \mathbb{N} \) and \( \Omega \subset \mathbb{R}^3 \). Let \( \Gamma_{\Omega,n}^{k_2} := \Gamma_{\Omega}^{k_2(\Omega)^n} \) and let
\[
(\Gamma_{\Omega,n,1}^{k_2+1}) := \{ v \mid v = \nabla w \ (\exists w \in \Gamma_{\Omega,1}^{k_2+1}) \}.
\] (321)

By diagonal argument possibly passing to a subsequence there exists an increasing sequence \( \{K_j\} \subset \mathbb{R}^3 \) of compact sets such that
\[
\|(\partial_t^{k_1} u_j(t), \partial_t^{k_1} P_j(t) - (\partial_t^{k_1} u(t), \partial_t^{k_1} P(t))\|_{\Gamma_{\Omega,n}^{k_2+1} \times (\Gamma_{\Omega,n}^{k_2+1})} \to 0 \quad (j \to \infty).
\] (322)

By assumption passing to a subsequence there exists \( c \geq 0 \) such that
\[
\|(\partial_t^{k_1} u_j(t), \partial_t^{k_1} P_j(t))\|_{\Gamma_{\Omega,n}^{k_2+1} \times (\Gamma_{\Omega,n}^{k_2+1})} \to c \quad (j \to \infty).
\] (323)

On the other hand, since \( \|(\partial_t^{k_1} u_j(t), \partial_t^{k_1} P_j(t))\|_{\Gamma_{\Omega,n}^{k_2+1} \times (\Gamma_{\Omega,n}^{k_2+1})} \) is bounded, for \((u', P') \in W_3 \times (W_1)\) there exists \( M_1 > 0 \) such that
\[
|< (\partial_t^{k_1} u_j(t), \partial_t^{k_1} P_j(t), (u'(t), P'(t)) >_{\Gamma^{k_2}_{(K_j,n)3} \times (\Gamma^{k_2+1}_{(K_j,n)3})} | \leq M_1 \|(u'(t), P'(t))\|_{\Gamma^{k_2}_{(K_j,n)3} \times (\Gamma^{k_2+1}_{(K_j,n)3})} \to 0 \quad (j \to \infty)
\] (324)

(325)

(326)
so that
\[
< (\partial_t^k u_j(t), \partial_t^k P_j(t)), (u'(t), P'(t)) >_{T^{k_{j,1}} \times (T^{k_{2,1}})} \quad (327)
\]
\[
-> < (\partial_t^k u(t), \partial_t^k P(t)), (u'(t), P'(t)) >_{T^{k_{j,1}} \times (T^{k_{2,1}})} \quad (328)
\]
\[
(j \to \infty).
\]
Thus
\[
|| (\partial_t^k u_j(t), \partial_t^k P_j(t)) - (\partial_t^k u(t), \partial_t^k P(t)) ||_{T^{k_{j,1}} \times (T^{k_{2,1}})}^2 \quad (330)
\]
\[
\to c^2 + || (\partial_t^k u(t), \partial_t^k P(t)) ||_{T^{k_{j,1}} \times (T^{k_{2,1}})}^2 \quad (331)
\]
\[
- 2 < (\partial_t^k u(t), \partial_t^k P(t)), (\partial_t^k u(t), \partial_t^k P(t)) >_{T^{k_{j,1}} \times (T^{k_{2,1}})} \quad (332)
\]
\[
(j \to \infty).
\]
Since
\[
|| (\partial_t^k u_j(t), \partial_t^k P_j(t)) - (\partial_t^k u(t), \partial_t^k P(t)) ||_{T^{k_{j,1}} \times (T^{k_{2,1}})}^2 \to 0 \quad (j \to \infty) \quad (334)
\]
it follows that
\[
c = || (\partial_t^k u(t), \partial_t^k P(t)) ||_{T^{k_{j,1}} \times (T^{k_{2,1}})}.
\]
Finally
\[
|| (\partial_t^k u_j(t), \partial_t^k P_j(t)) - (\partial_t^k u(t), \partial_t^k P(t)) ||_{T^{k_{j,1}} \times (T^{k_{2,1}})}^2 \quad (336)
\]
\[
\to c^2 + || (\partial_t^k u(t), \partial_t^k P(t)) ||_{T^{k_{j,1}} \times (T^{k_{2,1}})}^2 \quad (337)
\]
\[
- 2 < (\partial_t^k u(t), \partial_t^k P(t)), (\partial_t^k u(t), \partial_t^k P(t)) >_{T^{k_{j,1}} \times (T^{k_{2,1}})} \quad (338)
\]
\[
= 0 \quad (j \to \infty),
\]
and \( \{ q_j \} \) is convergent to 0 with respect to the ordinary topology. The assertion follows.

The set \( X \) equipped with the topology induced by \( p_k \) is, if exists, denoted by \( X^w \).

**Lemma 85.** Let \( \mathcal{X} \) be a sufficiently small convex neighbourhood of \( p \in \mathcal{U} \) with respect to the ordinary topology identified with the corresponding neighbourhood of \( 0 \in T_p \mathcal{U} \) (\( q \in \mathcal{X} \)). Then for any sequence \( \{ q_j \} \subset \mathcal{X}^w \) convergent to \( 0 \in \mathcal{X}^w \) it is convergent to \( 0 \in \mathcal{X} \) with respect to the ordinary topology.
Proof. Observe that by Lemma 70 $\Gamma^\infty(\mathbb{R}^3) = H^\infty(\mathbb{R}^3)$. Assume possibly after passing to a subsequence that $q_j \not\in c\mathcal{X}_p$ ($0 < c < 1$) for large $j$. $\mathcal{X}_p$ is sufficiently small so that by Lemma 84 there exists a subsequence $\{q_{j_k}\}$ convergent with respect to the ordinary topology. Let $W^1, \ldots, W^m$ be neighbourhoods of $p \in \mathcal{U}_p$ with respect to the topology induced by $p_k$ and $\bar{W}^1, \ldots, \bar{W}^m$ their closures with respect to this topology. Let $\mathcal{X}_p$ be the closure of $\mathcal{X}$ with respect to the ordinary topology. Then since $q_{j_k} \in \bar{W}^1 \cap \cdots \cap \bar{W}^m \cap (\mathcal{X}_p \setminus c\mathcal{X}_p)$ for large $l$ and $\bar{W}^1, \ldots, \bar{W}^m$ are arbitrary and since from above ($\mathcal{X}_p$ and so) $\mathcal{X}_p \setminus c\mathcal{X}_p$ is compact with respect to (the ordinary topology and so) the topology induced by $p_k$ an elementary argument shows that

$$\bigcap_W \left(W \setminus c\mathcal{X}_p\right) \neq \emptyset. \quad (341)$$

This contradicts with the definition of the topology of $\mathcal{U}_p$ induced by $p_k$. Thus $q_j \in c\mathcal{X}_p$ for large $j$. Since $c$ is arbitrary $\{q_j\} \subset \mathcal{X}_p$ is convergent to $0 \in \mathcal{X}_p$ with respect to the ordinary topology. The assertion follows. \qed

The integral with respect to the topologies induced by $p_k$ (resp. with respect to the ordinary topology of $\mathcal{U}_p$ and the topology of $\mathcal{V}_\Phi(p)$ induced by $p_k$) is denoted by $f$ (resp. $(f)^*$). See Definition 37.

Lemma 86. There exist a sufficiently small convex neighbourhood $\mathcal{X}_p$ of $p \in \mathcal{U}_p$ with respect to the ordinary topology and a sufficiently small neighbourhood of $0 \in (\mathcal{T}_\Phi(p)\mathcal{Y})^w$ identified with a sufficiently small neighbourhood $\mathcal{Y}_\Phi(p)$ of $\Phi(p) \in \mathcal{V}_\Phi^p$ such that the multi-valued map $q \in \mathcal{X}_p \mapsto \int_0^1 d\Phi(q') \in \mathcal{Y}_\Phi(p)$ has a continuous branch.

Proof. Since $\mathcal{X}_p$ is sufficiently small the map $d\Phi(q)$ ($q \in \mathcal{X}_p$) from $\mathcal{T}_\Phi(p)\mathcal{Y}$ to $\mathcal{Y}_\Phi(p)$ is continuous and the function $q \mapsto \int_0^1 ||d\Phi(q')||_{(\mathcal{T}_\Phi(p)\mathcal{Y})^w}$ is well-defined. Let $\gamma_q(t) := (1 - t)p + tq$ ($q \in \mathcal{X}_p$). Since $d\Phi$ is well-defined and a continuous map from $\mathcal{X}_p$ to $\mathcal{Y}_\Phi(p)$ there exists $M > 0$ such that

$$\lim_{h \to 0} \left|\int_{0}^{1} q(t + h) \right| d\Phi(q') \in \mathcal{(T}_\Phi(p)\mathcal{Y})^w. \quad (342)$$

$$= \lim_{h \to 0} \left|\int_{0}^{1} d\Phi((1 - t)\gamma_q(t_1) + t\gamma_q(t_1 + h))(\gamma_q(t_1 + h) - \gamma_q(t_1))dt \right|_{(\mathcal{T}_\Phi(p)\mathcal{Y})^w} \quad (343)$$

$$= \lim_{h \to 0} \left|\int_{0}^{1} d\Phi((1 - t)\gamma_q(t_1) + t\gamma_q(t_1 + h))(q - p)dt \right|_{(\mathcal{T}_\Phi(p)\mathcal{Y})^w} \quad (344)$$

$$\leq M. \quad (345)$$

By Theorem 39 there exists an integrable function $S_{\gamma_q}$ (see Definition 37) such that

$$\int_{0}^{1} d\Phi(q') \in \mathcal{Y}_\Phi(p). \quad (346)$$

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where $0 \leq t_1 \leq t_2 \leq 1$. Then by Lemma 85 an easy argument shows that the map $q \in \mathcal{X}^w_p \mapsto \int_0^1 S_{t_1}(t) dt \in \mathcal{Y}_{\Phi(p)}^w$ gives a desired branch. The assertion follows.

Observe that, where $\{W_r\}_r$ is a fundamental system of neighbourhoods of $0 \in \mathcal{U}_p$ with respect to the topology induced by $p_k$, a fundamental system of $0 \in \mathcal{X}^w_p$ is given by $\{W_r \cap \mathcal{X}^w_p\}_r$.

**Lemma 87.** There exist a sufficiently small convex neighbourhood $\mathcal{X}^w_p$ of $p \in \mathcal{U}_p$ with respect to the ordinary topology and a sufficiently small neighbourhood of $0 \in (T_{\Phi(p)}\mathcal{Y})^w$ identified with a sufficiently small neighbourhood $\mathcal{Y}^w_{\Phi(p)}$ of $\Phi(p) \in \mathcal{V}^w_{\Phi(p)}$ such that

\[
\Phi(q) = \Phi(p) + \int_p^q d\Phi(q') \quad (q \in \mathcal{X}^w_p, \Phi(q) \in \mathcal{Y}^w_{\Phi(p)})
\]  

holds for a well-defined continuous branch of $q \in \mathcal{X}^w_p \mapsto \int_p^q d\Phi(q') \in \mathcal{Y}^w_{\Phi(p)}$.

**Proof.** By Lemma 86 the multi-valued map $q \in \mathcal{X}^w_p \mapsto \int_p^q d\Phi(q') \in \mathcal{Y}^w_{\Phi(p)}$ has a continuous branch so that the function

\[
\Phi'(q) := \Phi(p) + \int_p^q d\Phi(q')
\]

has a well-defined continuous branch. Note that since $\Phi'$ has the derivative a.e. along $\gamma_q$ and since the topology of $\mathcal{X}^w_p$ induced by $p_k$ is weaker than the ordinary one it is obtained that (*) the derivatives of $\Phi'$ a.e. along $\gamma_q$ with respect to the both topologies at $q$ are equal to $d\Phi(q)$ a.e. in the path. On the other hand it is true that

\[
\Phi(q) = \Phi(p) + (\int_p^q)^* d\Phi(q').
\]  

Then $\Phi$ and $\Phi'$ satisfy the same conditions: (i) $\Xi(p) = \Phi(p)$ and (ii) the derivative of $\Xi(q)$ a.e. along $\gamma_q$ with respect to the ordinary topology of $\mathcal{X}_p$ and the topology of $\mathcal{Y}_{\Phi(p)}^w$ induced by $p_k$ coincides with $d\Phi(q)$ a.e. in the path. It follows that $\Phi(q) = \Phi'(q)$. By (*) it is obtained for $t \in [0, 1]$ that

\[
S_{\gamma_q}(t) = d\Phi(\gamma_q(t))\gamma_q(t)
\]  

(if LHS exists and is equal to $d(\Phi'(\gamma_q(t)))$). Thus for this branch

\[
\Phi(q) = \Phi(p) + \int_p^q d\Phi(q').
\]  

The assertion follows.

$\mathcal{X}^w_p$ and $\mathcal{X}^2_p$ are identified with the corresponding neighbourhoods of $0 \in T_q \mathcal{X}_p$ ($q \in \mathcal{X}^w_p$) and $0 \in T_Q T_q \mathcal{X}_p$ ($Q \in \mathcal{X}^2_p$).
Lemma 88. Shrink $\mathcal{X}_P$ if necessary. Then $d\Phi : (\mathcal{X}_P^w)^w \to (T\mathfrak{X}_\Theta(p))^w$ is continuous and the Fréchet derivative of $\Phi : \mathcal{X}_P^w \to \mathfrak{X}_\Theta^w(p)$ (see Definition 40) is equal to $d\Phi$ on $(\mathcal{X}_P^w)^w$.

Proof. As in Lemma 86 and Lemma 87 replacing $\Phi$ with $d\Phi$ it is proved that $d\Phi$ is continuous on $(\mathcal{X}_P^w)^w$. From this and Lemma 87 the assertion follows easily. □

Lemma 89. Shrink $\mathcal{X}_P$ if necessary. Then $d(d\Phi) : (\mathcal{X}_P^2)^w \to (TT\mathfrak{X}_\Theta(p))^w$ is continuous and the Fréchet derivative of $d\Phi : (\mathcal{X}_P^2)^w \to (T\mathfrak{X}_\Theta(p))^w$ (see Definition 40) is equal to $d(d\Phi)$ on $(\mathcal{X}_P^2)^w$.

Proof. This is proved as in Lemma 88. □

Lemma 90. Let $L((T_p\mathcal{X}_P)^w,(T\mathfrak{X}_\Theta(p))^w)$ be the set of continuous linear operators from $(T_p\mathcal{X}_P)^w$ to $(T\mathfrak{X}_\Theta(p))^w$. Then there exists $M' > 0$ such that

$$||d\Phi(q') - d\Phi(q)||_{L((T_p\mathcal{X}_P)^w,(T\mathfrak{X}_\Theta(p))^w)} \leq M'||q' - q||\mathcal{X}_P^w, \quad (352)$$

for $q, q', q' - q \in \mathcal{X}_P^w$, and such that

$$||\Phi(q') - \Phi(q) - d\Phi(q)(q' - q)||_{\mathfrak{X}_\Theta^w} \leq M'||q' - q||2\mathcal{X}_P^w, \quad (353)$$

for $q, q', q' - q \in \mathcal{X}_P^w$.

Proof. By Lemma 89 $d(d\Phi) : (\mathcal{X}_P^2)^w \to (TT\mathfrak{X}_\Theta(p))^w$ is continuous. Shrinking $\mathcal{X}_P$ if necessary there exists $M' > 0$ such that for $Q \in (\mathcal{X}_P^2)^w$

$$||d(d\Phi)(Q)|| \leq M'. \quad (354)$$

Let $\gamma(t) := (1 - t)q + tq'$ $(t \in [0, 1])$. Then, by Lemma 89

$$||d\Phi(q') - d\Phi(q)||_{L((T_p\mathcal{X}_P)^w,(T\mathfrak{X}_\Theta(p))^w)} = ||\int_0^1 \frac{\partial}{\partial t} d\Phi(\gamma(t))dt||_{L((T_p\mathcal{X}_P)^w,(T\mathfrak{X}_\Theta(p))^w)} \leq M'||q' - q||\mathcal{X}_P^w, \quad (355)$$

and

$$||\Phi(q') - \Phi(q) - d\Phi(q)(q' - q)||_{\mathfrak{X}_\Theta^w} = ||\int_0^1 \frac{\partial^2}{\partial t^2} \Phi(\gamma(t))dt||_{\mathfrak{X}_\Theta^w} \leq M'||q' - q||2\mathcal{X}_P^w, \quad (356)$$

The assertion follows. □
Take the completions $\mathcal{U}_p^k$ and $\mathcal{V}_{\Phi(p)}^k$ of $\mathcal{X}_p^w$ and $\mathcal{Y}_{\Phi(p)}^w$ induced from $p_k$. By Lemma 90, shrinking $\mathcal{U}_p^k$ and $\mathcal{V}_{\Phi(p)}^k$ if necessary, $d\Phi$ extends to a continuous map, $\Phi$ extends to a $C^1$-map from $\mathcal{U}_p^k$ to $\mathcal{V}_{\Phi(p)}^k$ and the Fréchet derivative with respect to the topologies of $\mathcal{U}_p^k$ and $\mathcal{V}_{\Phi(p)}$ of the extended $\Phi$ at $q$ is equal to the value $d\Phi(q)$ at $q$ of the extended $d\Phi$. $d\Phi(p)$ extends to a topological isomorphism between the completions $T_p\mathcal{U}_p^k$ and $T_{\Phi(p)}\mathcal{V}_{\Phi(p)}^k$ of $(T_p\mathcal{X})^w$ and $(T_{\Phi(p)}\mathcal{X})^w$ induced from $p_k$ so that since $\mathcal{X}_p^k$ is sufficiently small $d\Phi(q)$ ($q \in \mathcal{U}_p^k$) is a topological isomorphism. By Theorem 12 there exist sufficiently small neighbourhoods $U_p$ and $V_{\Phi(p)}(p \in \mathcal{U}_p^k)$ and $\Phi(p) \in \mathcal{V}_{\Phi(p)}^k$ such that the extended

$$\Phi : U_p \xrightarrow{\cong} V_{\Phi(p)}$$

is an isomorphism.

**Remark 91.** We have taken $O = \mathcal{U}_p^k$, $X$ the linear hull of $\mathcal{U}_p^k$, $Y$ the linear hull of $\mathcal{V}_{\Phi(p)}^k$, $\xi = \Phi$ and $a_0 = p$.

Observe that $V_{\Phi(p)}$ is an open set of the completion of $\mathcal{X}$. We thus obtain:

**Lemma 92.** $V_{\Phi(p)} (p \in \mathcal{X})$ form a Banach manifold.

Replacing $I$ with an interval $J := [t_0, t_0 + T_0]$ ($0 \leq t_0 < T$ and $T_0 > 0$ is small) we define $\mathcal{X}^J$, $\mathcal{Y}^J$, $U_p^J$, $\Phi^J$ etc. in the same way as $\mathcal{X}$, $\mathcal{Y}$, $U_p$, $\Phi$ etc. The above two (1 and 2) of Navier-Stokes conditions also hold if we replace $I$ with $J$ (see Remark 108).

**Remark 93.** Navier-Stokes condition 3 (see Remark 108) is satisfied.

We need a lemma.

**Lemma 94.** If $U_{p_1}^J \cap U_{p_2}^J \neq \phi$, it is open in $U_{p_1}^J$.

**Proof.** Let $p \in U_{p_1}^J \cap U_{p_2}^J \cap \mathcal{X}^J$. The tangent space $T_pU_{p_1}^J$ of $U_{p_1}^J$ at $p$ and the tangent space $T_pU_{p_2}^J$ of $U_{p_2}^J$ at $p$ are by definition given as the completions of $T_p\mathcal{X}^J$ and $\Phi^J$ is a local diffeomorphism on $U_{p_1}^J$ and $U_{p_2}^J$. Thus $T_pU_{p_1}^J = T_pU_{p_2}^J = (d\Phi^J(p))^{-1}(\text{completion of } T_{\Phi(p)}(\mathcal{Y}^J))$, where the completion of $T_{\Phi(p)}(\mathcal{Y}^J)$ is induced from $\mathcal{Y}^J \subset \bigcup V_{\Phi(p)}^J$. There exists a canonical isomorphism $\varphi_{p}^J|_W$ from a neighbourhood $W$ of $p \in \mathcal{X}^J$ to the corresponding neighbourhood $W'$ of $0 \in T_p\mathcal{X}^J$ and $U_{p_1}^J \cup U_{p_2}^J$ is the completion of $\mathcal{X}_{p_1}^J \cup \mathcal{X}_{p_2}^J$ induced from $p_k$. Extend $\varphi_{p}^J|_W$ to $W'$ (resp. $W''$), where $W'$ (resp. $W''$) is the closure of $W$ in $U_{p_1}^J$ (resp. $U_{p_2}^J$), to obtain a homeomorphism $\varphi_{p,1}^J$ (resp. $\varphi_{p,2}^J$). By definition $\varphi_{p,1}^J(W')$ (resp. $\varphi_{p,2}^J(W'')$) is the closure of $W'$ in $T_pU_{p_1}^J$ (resp. $T_pU_{p_2}^J$). Since $T_pU_{p_1}^J = T_pU_{p_2}^J$ it follows that $\varphi_{p,1}^J(W') = \varphi_{p,2}^J(W'')$ and since $U_{p_1}^J \cup U_{p_2}^J \subset U_{p_1}^J \cup U_{p_2}^J$, from the above, an easy argument shows that $(\varphi_{p,1}^J)^{-1} = (\varphi_{p,2}^J)^{-1}$. In particular $(\varphi_{p,2}^J)^{-1}((W'')^c)$ is open in $U_{p_1}^J$. By the definition of the topology $U_{p_1}^J \cap U_{p_2}^J \cap \mathcal{X}^J$ is dense in $U_{p_1}^J \cap U_{p_2}^J$.

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Let \( p' \in U^J_{p_1} \cap U^J_{p_2} \) be an inner point of \( U^J_{p_1} \). Take a small neighbourhood \( N \) of \( p' \) in \( U^J_{p_2} \) and consider \( U^J_{p_1} \cap N \cap \mathcal{X}^J \), which is by the same argument as above open in \( U^J_{p_1} \cap \mathcal{X}^J \), that is, of the form \( O \cap U^J_{p_1} \cap \mathcal{X}^J \), where \( O \) is an open set in \( U^J_{p_1} \). Take the closure of \( U^J_{p_1} \cap N \cap \mathcal{X}^J \). Then the set of all inner points of this closure is open in \( U^J_{p_1} \) and the boundary of the closure does not contain \( p' \). Hence \( U^J_{p_1} \cap U^J_{p_2} \) is open in \( U^J_{p_1} \). The assertion follows. \( \square \)

**Corollary 95.** \( U^J_p \) \((p \in \mathcal{X}^J)\) form a Banach manifold.

We shall prove the local existence and uniqueness of a sufficiently smooth solution of equation (277) (of which smoothness depends on \( k \in \mathbb{N} \)).

Let \( t_0 \in I \). Introduce to the inductive limits \( \mathcal{U}_{t_0} \) of \( \bigcup_p U^J_{p} \) for \( J \ni t_0 \) and \( \mathcal{C}_{t_0} \) of

\[
\mathcal{X}^{(k)}_{\mathcal{J}} := \{(f, r_2, u_0) \in (W^{k,k}_3)^J \times ((W^{k,k+2}_3)^J)_{\text{div}} \times \Gamma^{k+2}(\mathbb{R}^3)^3 | \nabla \cdot u_0 = r_2(t_0)\} \tag{362}
\]

\[
\mathcal{X}^{(k+2)}_{\mathcal{J}} := \{(f, r_2, u_0) \in \Gamma^{k+2}(\mathbb{R}^3)^3 | \nabla \cdot u_0 = r_2(t_0)\} \tag{363}
\]

for \( J \ni t_0 \) the natural topologies (the quotient topologies induced from the maps \( \prod_J U^J_{p} \to \mathcal{U}_{t_0} \) and \( \prod_J \mathcal{X}^{(k)}_{\mathcal{J}} \to \mathcal{C}_{t_0} \)). We first prove the following lemma.

**Lemma 96.** Let \( (f, r_2, u_0) \in \mathcal{X}^J \). There exists \( (u, \nabla p) \in \mathcal{X}^J \) such that

\[
\begin{cases}
(\dot{u} + (u \cdot \nabla)u - \nu \Delta u + \nabla p)|_{t=t_0} = f(t_0), \\
(\nabla \cdot u)|_{t=t_0} = r_2(t_0), \\
u|_{t=t_0} = u_0, \\
(\nabla \cdot \dot{u})|_{t=t_0} = \dot{r}_2(t_0).
\end{cases} \tag{364}
\]

**Proof.** Since

\[
(\dot{u} + (u \cdot \nabla)u - \nu \Delta u + \nabla p)|_{t=t_0} = f(t_0),
\]

by assumption

\[
-\dot{u}|_{t=t_0} - \nabla p(t_0) = (u_0 \cdot \nabla)u_0 - \nu \Delta u_0 - f(t_0). \tag{366}
\]

By \( (\nabla \cdot \dot{u})|_{t=t_0} = \dot{r}_2(t_0) \), \( -\dot{u}|_{t=t_0} \) is determined up to \( \text{Ker}(\nabla \cdot \cdot) \). Then by Corollary 71 it is confirmed that there exists such \((u, \nabla p)\). The assertion follows. \( \square \)

\[ \{\Phi^J\} \] induces a map \( \Phi_{t_0} : \mathcal{U}_{t_0} \to \mathcal{C}_{t_0} \).

**Lemma 97.** The induced map \( \Phi_{t_0} : \mathcal{U}_{t_0} \to \mathcal{C}_{t_0} \) is a homeomorphism.
Proof. $\Phi^J : \bigcup_p U_p^J \rightarrow \bigcup_p V_p^J$ is a local diffeomorphism so that by definition $\tilde{\Phi}_{t_0} : \mathcal{U}_{t_0} \rightarrow \mathcal{G}_{t_0}$ is a local homeomorphism. $\mathcal{U}_{t_0}$ is connected (since $\mathcal{X}^J$ is connected and each connected component of $U_p^J$ intersects with $\mathcal{X}^J$) and $\mathcal{G}_{t_0}$ is simply connected. We claim that $\tilde{\Phi}_{t_0}$ is surjective. Then $\tilde{\Phi}_{t_0}$ is a homeomorphism. By Lemma 96 it is proved that for any $r := (f, r_2, u_0) \in \mathcal{X}^J$ there exists a function $q' \in \bigcup_p U_p^J$ such that $\Phi^J(q')_{|t=t_0} = r(t_0)$, where $r^*(t) := (f^*(t), r_2^*(t), u_0^*)$ for $r^* := (f^*, r_2^*, u_0^*) \in \mathcal{X}^J$. Since $\Phi^J$ is an open map, there exists a deformation $q \in \bigcup_p U_p^J$ of $q'$ such that $\Phi^J(q)_{|t} = r(t)$ on a neighbourhood of $t_0$. More precisely there exists sufficiently small $\epsilon > 0$ such that any $r^* \in \mathcal{X}^J$ with $d(r^*, r(t_0))_{\div} := \Gamma^\infty(\mathbb{R}^3, \mathbb{R}^3) \times (\Gamma^\infty(\mathbb{R}^3, \mathbb{R}^3))_{\div, \div} < \epsilon$ (distance) is in $\text{Im} \Phi^J$. Let $d(r(t), \Phi^J(q')_{|t})_{\div, \div} := \{v \mid v = \nabla \cdot w \ (\exists w \in \Gamma^\infty(\mathbb{R}^3, \mathbb{R}^3))\}$. (368)

It is obtained for small $t_1 > 0$, $d(r(t), \Phi^J(q')_{|t})_{\div, \div} := \{v \mid v = \nabla \cdot w \ (\exists w \in \Gamma^\infty(\mathbb{R}^3, \mathbb{R}^3))\}$. (369)

(distance). By formula (367) and formula (369) there exists $q \in \bigcup_p U_p^J$ such that $\Phi^J(q)_{|t} = r(t)$ $(t_0 \leq t < t_0 + t_1)$. So the image of $\tilde{\Phi}_{t_0}$ contains the inductive limit $\mathcal{G}_{t_0}^\infty$ of $\mathcal{X}^J$ for $J \ni t_0$. Observe that the completion of $\mathcal{X}^J$ in $\mathcal{X}^{(k), J}$ coincides with the latter space so that the completion of $\mathcal{G}_{t_0}^\infty$ in $\mathcal{G}_{t_0}$ does with $\mathcal{G}_{t_0}$. From these the assertion is confirmed. Thus $\tilde{\Phi}_{t_0}$ is a homeomorphism.

Remark 98. Navier-Stokes condition $\mathcal{J}$ (see Remark 108) is satisfied.

We obtain a bijection $\tilde{\Phi} : \mathcal{U} := \coprod_{t_0} \mathcal{U}_{t_0} \rightarrow \mathcal{G} := \coprod_{t_0} \mathcal{G}_{t_0}$ and introducing a sheaf structure to $\mathcal{G}$ induced from $\mathcal{U}$ through this bijection a sheaf isomorphism, where the topology of $I$ is generated by $\mathcal{J}$'s. We thus conclude as follows.

Corollary 99. $\tilde{\Phi} : \mathcal{U} \rightarrow \mathcal{G}$ is a sheaf isomorphism.

Let $(f, r_2, u_0) \in \mathcal{X}^{(k), J}$. Then there exists a section $\tilde{p} := (u, \nabla p)$ of $\mathcal{U}$ such that $\tilde{\Phi}(\tilde{p}) = (f, r_2, u_0)$ on $I_1$, where $I_1 := [0, T_1)$ $(0 < T_1 \leq T)$ (or $I_1 := [0, T]$) is maximal and $\tilde{p}$ is locally unique in this topology of $I_1$ because $\tilde{\Phi}$ is an isomorphism. Hence $\tilde{p}$ is unique. Consider the following equation on $I_1$:

$$
\begin{bmatrix}
\frac{\partial u}{\partial t} + (u \cdot \nabla) u - \nu \Delta u - \nabla p \\
\nabla \cdot u \\
u \cdot u
\end{bmatrix}_{|t=0} := \Phi((u, \nabla p)) = 
\begin{bmatrix}
f \\
r_2 \\
u_0
\end{bmatrix}.
$$

We say $(u, \nabla p)$ satisfies equation (277) on $I_1$ if it satisfies equation (370). Now we proved the following.

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Theorem 100. Let $\nu > 0$. Let $k \in \mathbb{N}$. Let $I := [0, T]$ $(T > 0)$. Let
\[
(f, r_2, u_0) \in W^{k,k}_3 \times (W^{k,k+2}_{3})_{\text{div}} \times \Gamma^{k+2}(\mathbb{R}^3)^3
\]
such that $\nabla \cdot u_0 = r_2(0)$. Then there exists a unique $(u, \nabla p)$ satisfying equation (277) on $I_1 := [0, T_1]$ $(0 < T_1 \leq T)$ (or on $I_1 := [0, T]$) and that $I_1$ is maximal.

We introduced to $\mathcal{Y}$ the relative topology induced from (300) (which depends on $k$) and defined the Banach manifold $\bigcup U_p$ in Corollary 95 and the sheaf $\mathcal{Y}$ after the proof of Lemma 97. Now we remark the following:

Remark 101. In fact $(u, \nabla p) \in \mathcal{Y}(I_1)$.

We are going to prove Theorem 78. That $\mathcal{Y} = \mathcal{Z}$ is proved by Navier-Stokes conditions (see Remark 108). Since $(f, 0, u_0) \in A$ and $\mathcal{Y} = \Phi(\mathcal{X})$ there exists $p \in \mathcal{X}$ such that $\Phi(p) = (f, 0, u_0)$, which defines a smooth solution of equation (277). We formalize this in the following way.

Let $\Omega \subset \mathbb{R}^3$ be a subset. Let
\[
W_{\Omega,n} := W_{\Omega,n}^\infty,
\]
\[(W_{\Omega,1})_{\text{var}} := \{v \mid v = \nabla w \ (\exists w \in W_{\Omega,1})\},\]
\[(W_{\Omega,3})_{\text{div}} := \{v \mid v = \nabla \cdot w \ (\exists w \in W_{\Omega,3})\},\]
\[\mathcal{X}_\Omega := W_{\Omega,3} \times (W_{\Omega,1})_{\text{var}}.
\]

Lemma 102. Let $(f, r_2, u_0) \in \mathcal{Z}$. Let $x \in \mathbb{R}^3$. Then there exists $(u^x, \nabla p^x) \in \mathcal{X}_x$ satisfying equation (277) on $I \times \{x\}$.

Proof. The assertion easily follows.

Lemma 103. Let $(f, r_2, u_0) \in \mathcal{Z}$. Let $x \in \mathbb{R}^3$. Then there exist a compact neighbourhood $K_x$ of $x$ and $(u^{K_x}, \nabla p^{K_x}) \in \mathcal{X}_{K_x}$ satisfying equation (277) on $I \times K_x$.

Proof. By Lemma 102 it follows that there exists $(u^x, \nabla p^x) \in \mathcal{X}_x$ satisfying equation (277) on $I \times \{x\}$. Extend it arbitrarily to $I \times \mathbb{R}^3$ to obtain $(u', \nabla p') \in \mathcal{Z}$. Note that
\[
d((f, r_2, u_0), \Phi((u', \nabla p')))_{W_{(x),3} \times (W_{(x),3})_{\text{div}} \times \Gamma^{\infty}(\{z\})^3}
\]
(distance) is small for any $z$ around $x$. Then since by Lemma 30 $\Phi : \mathcal{Z} \rightarrow \mathcal{Y}$ is an open map there exist a compact neighbourhood $K_x$ of $x$ and $(u^{K_x}, \nabla p^{K_x}) \in \mathcal{X}_{K_x}$ satisfying equation (277) on $I \times K_x$. The assertion follows.

Lemma 104. Let $(f, r_2, u_0) \in \mathcal{Z}$. Then there exist a closed neighbourhood $K_\infty$ of $\infty$ and $(u^{K_\infty}, \nabla p^{K_\infty}) \in \mathcal{X}_{K_\infty}$ satisfying equation (277) on $I \times K_\infty$. 

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Proof. By definition the limits of the derivatives of an element of $\Gamma^\infty(\mathbb{R}^3)$ as $x \to \infty$ are 0 so that there exists $(u', \nabla p') \in \mathcal{X}$ satisfying equation (277) on $I \times \{\infty\}$. Then there exists a neighbourhood $K_\infty$ of $\infty$ such that

$$d((f,r_2,u_0), \Phi((u', \nabla p')))_{W_{K_\infty,3}(\mathcal{X})} < \epsilon$$

(distance) is small. By Lemma 30 it is obtained that $\Phi : \mathcal{X} \to \mathcal{Y}$ is a homeomorphism. The assertion follows.

**Remark 105.** Navier-Stokes condition 5 (see Remark 65) is satisfied.

**Lemma 106.** $\mathcal{Y} = \mathcal{X}$.

Proof. Let $(f,r_2,u_0) \in \mathcal{X}$. By Lemma 103 there exists a family $\{(u^{K,\infty}, \nabla p^{K,\infty})\}_x$ such that each $(u^{K,\infty}, \nabla p^{K,\infty})$ satisfies equation (277) on $I \times K_x$. By Lemma 104 there exists $(u^{K,\infty}, \nabla p^{K,\infty}) \in \mathcal{X}$ satisfying equation (277) on $I \times K_\infty$. Since $\mathbb{R}^3 \setminus K_\infty$ is compact we obtain a finite set $\{\infty\} \cup \{x_\lambda\}$ such that each $K_x$ intersects with another (and $K_\infty$) in a set of Lebesgue measure 0 and $K_\infty \cup \bigcup_{\lambda} K_{x_\lambda} = \mathbb{R}^3$. It follows that there exists $(u, \nabla p) \in (L^2(I \times \mathbb{R}^3)^3)^2$ that is smooth a.e. and satisfies equation (277) a.e. By Lemma 30 and Theorem 100 $\Phi : \mathcal{X} \to \mathcal{Y}$ is a homeomorphism. Extend $\Phi$ to a continuous map from $\mathcal{Y}(\subset \mathcal{X})$ to

$$\{(u, \nabla p) \in (L^2(I \times \mathbb{R}^3)^3)^2 : u, \nabla p \text{ are smooth a.e.}\}. \quad (378)$$

Since the set $\{\infty\} \cup \{x_\lambda\}$ is finite, considering the convolutions $(u_x, \nabla p_x)$ with mollifiers it is proved that $(u, \nabla p) \in \Phi^{-1}(\mathcal{Y})$ and thus

$$(u, \nabla p) = \Phi^{-1}((f,r_2,u_0)) \quad (379)$$

(note that $(u_x|_{K_x}, \nabla p_x|_{K_x}) (z = x_\lambda, \infty)$ is convergent in $\mathcal{X}_{K_x}$). Let $(s,y) \in I \times \mathbb{R}^3$ be a singular point of $(u, \nabla p)$. Take another finite decomposition $K_y \cup K_\infty \cup \bigcup_{\mu} K_{x_\mu} = \mathbb{R}^3$ and construct $(u_1, \nabla p_1)$ that is smooth at $(s,y)$ and satisfies equation (277) a.e. Since

$$(u, \nabla p) = \Phi^{-1}((f,r_2,u_0)) = (u_1, \nabla p_1) \quad (380)$$

it follows that $(u, \nabla p)$ is smooth at $(s,y)$. This contradiction shows that $(u, \nabla p) \in \mathcal{X}$. Since

$$(f,r_2,u_0) = \Phi((u, \nabla p)) \in \mathcal{Y} \quad (381)$$

it is concluded that $\mathcal{X} \subset \mathcal{Y}$. Since the other inclusion is trivial the assertion follows.

**Remark 107.** Navier-Stokes conditions (see Remark 108) are used.
Proof of Theorem 78. Since \((f, 0, u_0) \in \mathcal{Z} = \mathcal{Y}\) (see Lemma 106) there exists \(p \in \mathcal{X}\) such that \(\Phi(p) = (f, 0, u_0)\), which defines a solution of (277). Further since \(\Phi: \mathcal{X} \to \mathcal{Y}\) is a homeomorphism the solution is unique. The assertion follows. \(\square\)

Remark 108. In the proof of Theorem 78 we used the following Navier-Stokes conditions (which are not assumptions) and conclude \(\Phi\) is a homeomorphism.

1. \(\Phi: \mathcal{X} \to \mathcal{Y}\) is a \(C^\infty\)-map such that \(d\Phi(p): T_p\mathcal{X} \to T_{\Phi(p)}\mathcal{Y}\) is a linear isomorphism.

2. Each seminorm \(p_k\) of \(\mathcal{Y}\), which is induced from the relative topology from

\[
W_3^{k,k} \times (W_3^{k,k+2})_{\text{div}} \times \Gamma^{k+2}(\mathbb{R}^3)^3,
\]

is actually a norm, that is, it satisfies the condition that \(p_k(r) = 0\) implies \(r = 0\).

3. The above holds if we replace \(I\) with \(J\).

4. For any \(r \in \mathcal{Z}\) there exists \(q \in \bigcup_p U_p^J\) such that \(\Phi^J(q)|_t = r(t)\) \((t_0 \leq t < t_0 + t_1)\) for small \(t_1\).

5. Let \((f, r_2, u_0) \in \mathcal{Z}^2\). Let \(x \in \mathbb{R}^3\). Then there exist a compact neighbourhood \(K_x\) of \(x\) and \((u^{K_x}, \nabla p^{K_x}) \in \mathcal{X}_{K_x}\) satisfying equation (277) on \(I \times K_x\). (Further there exist a closed neighbourhood \(K_\infty\) of \(\infty\) and \((u^{K_\infty}, \nabla p^{K_\infty}) \in \mathcal{X}_{K_\infty}\) satisfying equation (277) on \(I \times K_\infty\).

12 Inequality

Theorem 109. Let \(\nu > 0\). Let \(I := [0, \infty)\). Let \(u_0 \in \Gamma^\infty(\mathbb{R}^3)^3\) such that \(\nabla \cdot u_0 = 0\). Assume further

\[
|\partial_x^\alpha u_0(x)| \leq C_{\alpha,K} (1 + |x|)^{-K}
\]

for each nonnegative integer \(K\) and each multi-index \(\alpha\). Then there exists an unique \((u, \nabla p) \in W_3 \times (W_1)^3\) satisfying the equation

\[
\begin{align*}
\frac{\partial u}{\partial t} &= -(u \cdot \nabla)u + \nu \Delta u - \nabla p, \\
\nabla \cdot u &= 0, \\
|u|_{t=0} &= u_0, \\
\lim_{|x| \to \infty} |\partial_x^\alpha u(x, t)| &= 0 \quad (\forall t, \alpha).
\end{align*}
\]

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and \( \gamma > 0 \), such that for any \( t \in I \)
\[
\int_{\mathbb{R}^3} |u(t,x)|^2 \, dx \leq \gamma. \tag{385}
\]

**Proof.** Observe that \( u_0 \in \Gamma^\infty(\mathbb{R}^3)^3 \). By Theorem 78 there exists an unique solution of equation (384) on \([0,T]\) for any \( T > 0 \) and patching the solutions there exists an unique global solution \((u, \nabla p) \in W_3 \times (W_1)_\nabla\) of the equation on \( I \). Then,
\[
\frac{d}{dt} < u, u >_{L^2(\mathbb{R}^3)^3} = 2 < \frac{\partial u}{\partial t}, u >_{L^2(\mathbb{R}^3)^3}
\]
\[
= 2 < (u \cdot \nabla)u + \nu \Delta u - \nabla p, u >_{L^2(\mathbb{R}^3)^3}
\]
\[
= 2 \int_{\mathbb{R}^3} (\text{div}\, u) \frac{|u|^2}{2} \, dx + 2 \nu < \Delta u, u >_{L^2(\mathbb{R}^3)^3} + 2 \int_{\mathbb{R}^3} p(\text{div}\, u) \, dx
\]
\[
= 2 \nu < \Delta u, u >_{L^2(\mathbb{R}^3)^3} \leq 0.
\]
Thus for any \( t \in I \)
\[
0 \leq ||u(t, \cdot)||^2_{L^2(\mathbb{R}^3)^3} \leq ||u_0||^2_{L^2(\mathbb{R}^3)^3} < \infty. \tag{392}
\]
It follows that there exists \( \gamma > 0 \) such that for any \( t \in I \)
\[
\int_{\mathbb{R}^3} |u(t,x)|^2 \, dx \leq \gamma. \tag{393}
\]
The assertion follows. \( \square \)

### 13 CA-concepts

**Definition 110.** Types are defined by the following.

1. \( i, o \) are types;
2. Assume \( \tau_1, \ldots, \tau_n \) are (finitely many) types and then \((\tau_1, \ldots, \tau_n)\) is a type;
3. Assume \( \tau_1, \ldots, \tau_n, \sigma \) are (finitely many) types and then \((\tau_1, \ldots, \tau_n) \rightarrow \sigma\) is a type.

**Definition 111.** CA-alphabet consists of the following.

1. **Constant symbols**
   \( c^i_\lambda, M_\lambda, (\lambda \in \Lambda \text{ and } \Lambda \text{ is a finite set}), \)
   \( M_\lambda^{N, \alpha, \lambda}, R_{\lambda, \delta, \lambda} \) (\( \theta^\lambda \in \Theta^\lambda \) and \( \Theta^\lambda \) is a set),
   \( (M_\lambda, R_{\lambda, \delta, \lambda})_{\lambda \in \Lambda}, \prod[M_\lambda], (M_\lambda, R_{\lambda, \delta, \lambda})_{\lambda \in \Lambda}, \)
   \( \leq_\delta \) (\( \delta \in \Delta \) and \( \Delta \) is a set);
(2) individual variables $x_0^\tau, x_1^\tau, \ldots$;
(3) predicate symbols $\in\tau, \neq\tau, T\tau, X\tau$;
(4) function symbols (of type $(\tau_{a_i}) \rightarrow \sigma (\tau_{a_j}, \sigma \neq o)$) $\bar{f}_j$ ($j \in J$ and $J$ is a set);
(5) logical symbols $\neg, \land, \lor, \rightarrow, \leftrightarrow, \forall\tau, \exists\tau, \bot$;
(6) auxiliary symbols $()$, $[ ]$.

Remark 112. $\bar{f}_j$ is a function symbol of $A_j$ ($1 \leq A_j < \infty$) variables.

Definition 113. TERM is the smallest set $X$ with the following properties. An element of TERM is called a term.
(1) Constant symbols of type $\neq o$ and individual variables of type $\neq o$ are elements of $X$.
(2) $t_1, \ldots, t_{A_j} \in X$ are of type $\tau_1, \ldots, \tau_{A_j}$ and $\bar{f}_j$ is of type $(\tau_{a_j}) \rightarrow \sigma$ $\Rightarrow$ $\bar{f}_j((t_{a_j})) \in X$ is of type $\sigma$.

Definition 114. FORM is the smallest set $X$ with the following properties. An element of FORM is called a formula. (Formulas are said to be of type $o$.)
(1) $\bot \in X$;
constant symbols of type $o$ $\in X$;
individual variables of type $o$ $\in X$;
t_1, t_2 of type $\tau$ $\Rightarrow$ $(t_1 \in \tau, t_2) \in X$;
t_1, t_2 of type $\tau$ $\Rightarrow$ $(t_1 \neq \tau, t_2) \in X$;
t of type $\tau$ $\Rightarrow$ $T\tau(t), X\tau(t) \in X$;
s of type $(\tau_1, \ldots, \tau_n)$ and $t_1, \ldots, t_n$ of type $\tau_1, \ldots, \tau_n \Rightarrow s(t_1, \ldots, t_n) \in X$.

Note: These formulas are said to be atomic.
(2) $\varphi, \psi \in X \Rightarrow (\varphi \square \psi) \in X$ ($\square = \land, \lor, \rightarrow, \leftrightarrow$);
(3) $\varphi \in X \Rightarrow \neg \varphi \in X$;
(4) $\varphi \in X \Rightarrow \forall\tau x_1^\tau, \exists\tau x_1^\tau \varphi \in X$.

Definition 115. A formula is primitive if it is constructed from variables,
$\{\in\tau, \neq\tau, \land, \lor, \rightarrow, \leftrightarrow, \neg, \forall\tau, \exists\tau\}$. (Variables of type $o$ are not involved.)

Definition 116. The set $FV(\varphi)$ of free variables of a formula $\varphi$ and the set $FV(t)$ of free variables of a term $t$ are defined by the following.
(1) $FV(x_1^\tau) := \{x_1^\tau\}$;
$FV(c) := \phi$ if $c$ is a constant symbol;
(2) $FV(\bar{f}_j((t_{a_j}))) := \bigcup FV(t_{a_j})$;
(3) $FV(t_1 \neq \tau, t_2) := FV(t_1) \cup FV(t_2)$;
$FV(T\tau(t)) := FV(X\tau(t)) := FV(t)$;
$FV(s(t_1, \ldots, t_n)) := FV(s) \cup \bigcup FV(t_i)$;
$FV(\bot) := \phi$;
(4) $FV(\varphi \square \psi) := FV(\varphi) \cup FV(\varphi) \square = \land, \lor, \rightarrow, \lor \leftrightarrow$;
(5) $FV(\neg \varphi) := FV(\varphi)$;
(6) $FV(\forall\tau x_1^\tau \varphi) := FV(\exists\tau x_1^\tau \varphi) := FV(\varphi) \setminus \{x_1^\tau\}$. 

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Definition 117. A bounded variable is a variable that is not free.

Definition 118. A CA-structure \( M = (\{M_\lambda, R_{\alpha_\lambda}\}_{\lambda \in \Lambda}, \leq) \) consists of the following.
(1) sets
\( E_0 = \{0, 1\}; \)
\( E_\varphi (\neq \varphi); \)
\( E_\tau = \mathcal{P}(E_{\tau_1} \times \cdots \times E_{\tau_n}) \quad \tau = (\tau_1, \ldots, \tau_n) \) (power set);
(2) sets \( M_\lambda \quad \lambda \in \Lambda \) (\( \Lambda \) is a finite set);
\( E = \bigcup M_\lambda; \)
(3) a set \( =_\tau \subset E_\varphi \times E_\tau; \)
(4) a fixed set \( X_i \) with the following properties
(a) \( \) It is proved that \( X^0 \) exists uniquely in ZFC, where ZFC is a model of ZFC;
(b) \( X^0 \subset X_i; \)
(c) \( x \in X_i \land y \in x \rightarrow y \in X_i; \)
(d) \( X_i \) is the smallest set with the above two properties
sets
\( X_0 = \{0, 1\}; \)
\( X_\tau = \mathcal{P}(X_{\tau_1} \times \cdots \times X_{\tau_n}) \quad \tau = (\tau_1, \ldots, \tau_n); \)
(5) functions \( F_j \) \( (A_j \) variables, of type \( \tau_{a_j} \)) from \( \prod_{1 \leq a_j \leq A_j} (E_{\tau_{a_j}} \cup X_{\tau_{a_j}}) \)
to \( E_\sigma \cup X_\sigma; \)
(6) \( c^*_\tau \) elements in \( E_\tau; \)
\( M_\lambda \) elements in \( E_{(1)}; \)
\( M_\lambda^{N_{\tau_{a\lambda}}}; \) elements in \( E_{(t, \ldots, t)}; \)
\( R_{\lambda, a\lambda} \) elements in \( E_{(t, \ldots, t)}; \)
\( (M_\lambda, R_{\lambda, a\lambda})_{\lambda \in \Lambda}, \prod \{(M_\lambda) \text{ elements in } E_{(1)}; \}
\( (M_\lambda, R_{\lambda, a\lambda})_{\lambda \in \Lambda}, \leq \delta \) elements in \( E_{(t, \ldots, (t))}; \)
(7) \( \subseteq \tau \subset (E_\tau \cup X_\tau) \times (E_\tau \cup X_\tau) \) is a restriction of \( \subseteq \in \) in ZFC.

Remark 119. (1) Each \( M_\lambda \) is called an universe.
(2) Each \( R_{\lambda, a\lambda} \) is called a relation in \( M_\lambda. \)
(3) Each \( \leq \delta \) is called a relation among \( (M_\lambda, R_{\lambda, a\lambda})_{\lambda \in \Lambda}. \)

Definition 120. A closed formula or a sentence is a formula without free variables. A set \( \Gamma \) of axioms is a set of sentences.

Definition 121. Where \( t \) is a term of type \( \tau \), for a formula \( \varphi, \varphi[t/x^*_\tau] \) is defined by the following.
(1)
\[ y'_t[t/x^\tau_i] := \begin{cases} y'_t & \text{if } y'_t \neq x^\tau_i, \\ t & \text{if } y'_t = x^\tau_i, \end{cases} \tag{394} \]

where \( y'_t \) is a variable of type \( \tau' \), and
\[ c'^\tau[t/x^\tau_i] := c'^\tau, \tag{395} \]

where \( c'^\tau \) is a constant of type \( \tau' \).

(2) \( \bar{f}_i((t_{a_i}))[t/x^\tau_i] := \bar{f}_i((t_{a_i}[t/x^\tau_i])) \),

(3) \( \bot[t/x^\tau_i] := \bot; \)
\( T_\tau(t_1)[t/x^\tau_i] := T_\tau(t_1[t/x^\tau_i]); \)
\( X_\tau(t_1)[t/x^\tau_i] := X_\tau(t_1[t/x^\tau_i]); \)
\( (t_1 \equiv_\tau t_2)[t/x^\tau_i] := t_1[t/x^\tau_i] \equiv_\tau t_2[t/x^\tau_i]; \)
\( (t_1 \in_\tau t_2)[t/x^\tau_i] := t_1[t/x^\tau_i] \in_\tau t_2[t/x^\tau_i]; \)
\( s(t_1, \ldots, t_n)[t/x^\tau_i] := s[t/x^\tau_i](t_1[t/x^\tau_i], \ldots, t_n[t/x^\tau_i]); \)

(4) \( (\phi \land \psi)[t/x^\tau_i] := (\phi[t/x^\tau_i]) \land (\psi[t/x^\tau_i]) \)
\( (\neg \phi)[t/x^\tau_i] := \neg(\phi[t/x^\tau_i]); \)

(5)
\[
(\forall_{\tau'} \varphi)[t/x^\tau_i] := \begin{cases} \forall_{\tau'} \varphi[t/x^\tau_i] & \text{if } x^\tau_i \neq y^\tau', \\ \forall_{\tau} x^\tau_i \varphi & \text{if } x^\tau_i = y^\tau', \end{cases} \tag{396} \]
\[
(\exists_{\tau'} \varphi)[t/x^\tau_i] := \begin{cases} \exists_{\tau'} \varphi[t/x^\tau_i] & \text{if } x^\tau_i \neq y^\tau', \\ \exists_{\tau} x^\tau_i \varphi & \text{if } x^\tau_i = y^\tau'. \end{cases} \tag{397} \]

where \( t \) is of type \( \tau \).

**Remark 122.** We define \( \varphi[t/x^\tau_i] \) for a formula \( \varphi \) in the same way.

**Definition 123.** The language \( L(M) \) of a CA-structure \( M = ((M_\lambda, R_{\Lambda^\lambda}), \leq_\Lambda) \) consists of

- as predicate constants \( \equiv_\tau, \in_\tau, T_\tau, X_\tau; \)
- as function symbols \( \bar{f}_i; \)
- as constant symbols \( \bar{a} \) \( (a \in \mathcal{U}) \), \( c^\tau \), \( M_\lambda \);
- \( M^F_{\lambda,\mathcal{U},\Lambda}; (M^F_{\lambda,\mathcal{U},\Lambda})_{\lambda \in \Lambda}; (M_{\lambda, R_{\Lambda^\lambda}})_{\lambda \in \Lambda}; (M_{\lambda, R_{\Lambda^\lambda}})^F_{\lambda \in \Lambda}; \leq_\Lambda; \prod \{M_\lambda\}; \)

Here, \( \mathcal{U} \) is the universe of \( M \) as an ordinary structure. The language \( L \) consists of the same sets but constant symbols \( \bar{a} \), which are not constituents of \( L \).

**Definition 124.** A closed term is a term without free variables. The interpretation of the closed terms of \( L(M) \) is the following map \( (\cdot)^M : \text{TERM}_c \longrightarrow \mathcal{U}. \)

(1) \( c^M := c, \bar{a}^M := a; \)

(2) \( (\bar{f}_i((t_{a_i})))^M := F_i(t_{a_i}^M). \)
Definition 125. Let $SENT$ be the set of sentences. The interpretation of a sentence $\varphi$ of $L(M)$ in $M$ is the following map $\[\vdash_M : SENT \rightarrow \{0, 1\}$.

1. $\bot_M = 0$;
2. $[P]_M = P^M : P$ (is a constant of type $0$);
3. We denote $t^M = [t]_M$ for a sentence $t$;

\[
[t_1 \models t_2]_M := \begin{cases} 1 & \text{if } t_1^M \models t_2^M \text{ and } t_1^M, t_2^M \in E \cup X, \\ 0 & \text{otherwise}, \end{cases} \tag{398}
\]

\[
[t_1 \models t_2]_M := \begin{cases} 1 & \text{if } t_1^M \models t_2^M \text{ and } t_1^M, t_2^M \in E \cup X, \\ 0 & \text{otherwise}, \end{cases} \tag{399}
\]

\[
[T^M(t)]_M := \begin{cases} 1 & \text{if } t^M \models E, \\ 0 & \text{otherwise}, \end{cases} \tag{400}
\]

\[
[X^M(t)]_M := \begin{cases} 1 & \text{if } t^M \models X, \\ 0 & \text{otherwise}, \end{cases} \tag{401}
\]

\[
[s(t_1, \ldots, t_n)]_M := \begin{cases} 1 & \text{if } (t_1^M, \ldots, t_n^M) \in s^M, \\ 0 & \text{otherwise}, \end{cases} \tag{402}
\]

2. $[\varphi \land \psi]_M := \min([\varphi]_M, [\psi]_M)$;
3. $[\varphi \lor \psi]_M := \max([\varphi]_M, [\psi]_M)$;
4. $[\varphi \rightarrow \psi]_M := \max(1 - [\varphi]_M, [\psi]_M)$;
5. $[\varphi \leftrightarrow \psi]_M := 1 - ([\varphi]_M - [\psi]_M)$;

Def. 126. Let $\Lambda = \{\lambda_1, \ldots, \lambda_N\}$. $\Gamma'$ is defined by the following.

1. $\bot$;
2. $\forall x, y \forall \tau_1 \ldots \forall \tau_n \exists \lambda_1 \ldots \exists \lambda_n [x(z_1, \ldots, z_n) \leftrightarrow y(z_1, \ldots, z_n)] \rightarrow x \models y$, $(\tau = (\tau_1, \ldots, \tau_n))$;
3. $\forall a, y \forall \tau_1 \exists \lambda_1 \forall \tau_2 y(z_1, \ldots, \lambda_2) \rightarrow \varphi$, where $FV(\varphi) \subset \{x_1, \ldots, x_m, \lambda_1, \ldots, \lambda_n\}$ and $\varphi$ is a formula;
4. $\forall \tau_1 \exists \tau_2 \forall \tau_3 \ldots \forall \tau_n \exists \lambda_1 \ldots \exists \lambda_m [y(z_1, \ldots, \lambda_n) \leftrightarrow \varphi]$. 

\[\forall \tau_1 \exists \tau_2 \forall \tau_3 \ldots \forall \tau_n \exists \lambda_1 \ldots \exists \lambda_m [y(z_1, \ldots, \lambda_n) \leftrightarrow \varphi].\]
(9) \(\forall x_1 \ldots \forall x_{N_{\lambda^x}}\)
\((R_{\lambda^x}(x_1, \ldots, x_{N_{\lambda^x}}) \rightarrow M_{\lambda^x}(x_1, \ldots, x_{N_{\lambda^x}}));\)
(10) \(\forall x, x(T_1(x) \leftrightarrow \bigvee_j M_{\lambda_j}(x));\)
(11) \(\forall x, x(T_\tau(x) \leftrightarrow \bigvee_j M_{\lambda_j}(x));\)
(12) Let \(\varphi\) be atomic.
\(\forall x, y [x =_\tau y \rightarrow (\varphi[x/z] \rightarrow \varphi[y/z])]\) (\(z\) is an individual variable of type \(\tau\)).

**Definition 127.** For a formula \(\gamma\), \(\gamma^T\) is defined by the following.

1. \(\varphi^T := \varphi\)
   if \(\varphi\) is atomic;
2. \((\varphi \land \psi)^T := \varphi^T \land \psi^T\) \((\land = \land, \lor, \rightarrow, \leftrightarrow);\)
3. \((\neg \varphi)^T := \neg (\varphi)^T;\)
4. \((\forall x^T \varphi)^T := \forall x^T (T_\tau(x^T) \rightarrow \varphi^T);\)
5. \((\exists x^T \varphi)^T := \exists x^T (T_\tau(x^T) \land \varphi^T).\)

**Definition 128.** Let \(\Gamma\) be a set of sentences without \(X_\tau, \notin_\tau\). Then defining
\(\Gamma^T := \{\gamma^T \mid \gamma \in \Gamma\}\)
(403)
\(\Gamma^T \cup \Gamma'\) is denoted by \(\Gamma\) (in an abuse of notation).

**Definition 129.** A CA-structure \(\mathbb{M}\) is a CA-model of a sentence \(\varphi\) if
\([\varphi]_{\mathbb{M}} = 1.\)
(404)
In this case we write \(\mathbb{M} \models \varphi\). A CA-structure \(\mathbb{M}\) is a CA-model of the axioms \(\Gamma\) if
\(\varphi \in \Gamma \Rightarrow [\varphi]_{\mathbb{M}} = 1.\)
(405)
In this case we write \(\mathbb{M} \models \Gamma\). We write \(\models \varphi\) if "for any CA-structure \(\mathbb{M}\), \(\mathbb{M} \models \varphi\) holds." We write \(\Gamma \models \varphi\) (\(\varphi\) is a sentence) if
\(\mathbb{M} \models \Gamma \Rightarrow \mathbb{M} \models \varphi.\)
(406)
Definition 130. Where $t$ is a term or a formula and $x$ is an individual variable, $t$ is free for $x$ for a formula $\varphi$ if

1. $\varphi$ is atomic;
2. $\varphi := \varphi_1 \Box \varphi_2$ or $\neg \varphi_1 \Box = \land, \lor, \rightarrow, \leftrightarrow$ and $t$ is free for $x$ in $\varphi_1$ (and $\varphi_2$);
3. $\varphi := \exists_y \psi$ or $\forall_y \psi$, $y \notin FV(t)$ and $t$ is free for $x$ in $\psi$, where $x \neq y$.

Definition 131. We introduce the following derivation rules into formulas of a fixed language.

1. 
   \[
   \frac{\varphi(x^*_1)}{\forall \tau x^*_1 \varphi(x^*_1)}
   \]
   
   $x^*_1$ is not free in any assumption before $\varphi(x^*_1)$

2. 
   \[
   \frac{\forall \tau x^*_1 \varphi(x^*_1)}{\varphi(t)}
   \]
   
   $t$ is of type $\tau$ and free for $x^*_1$ in $\varphi$

3. 
   \[
   \frac{\varphi(t)}{\exists \tau \varphi(x)}
   \]
   
   $t$ is of type $\tau$ and $x$ is a variable of type $\tau$

4. 
   \[
   [\varphi] \vdash \exists \tau \varphi \psi \vdash \psi
   \]

Remark 132. The other derivation rules are defined naturally.

Definition 133. Let $\Gamma$ be a set of formulas and $\varphi$ a formula. If $\varphi$ is obtained from $\Gamma$ by a finite number of applications of derivation rules, we say that there exists a derivation from $\Gamma$ to $\varphi$ and write

\[ \Gamma \vdash \varphi. \]

Definition 134. Let $\text{FORM}_{\text{prim}}$ be the set of primitive formulas, a (class) function $F' : \{ \varphi \mid \varphi \in \text{FORM}_{\text{prim}} \} \to \{ \text{well formed formulas} \}$ is defined by the following.

1. 
   \[ F'(x \in \tau y) := x \in y; \]
   \[ F'(s(t_1, \ldots, t_n)) := (t_1, \ldots, t_n) \in s, \]
   where $s, t_1, \ldots, t_n$ in LHS and those in RHS are appropriate variables;
2. 
   \[ F'(\varphi \Box \psi) := F'(\varphi) \Box F'(\psi) \Box = \land, \lor, \rightarrow, \leftrightarrow; \]
3. 
   \[ F'(\neg \varphi) := \neg F'(\varphi); \]
\( (4) \)
\[ F'(\forall \tau \varphi(x)) := \forall x(x \in \mathcal{X}_\tau \rightarrow F'(\varphi(x))); \]
\[ F'(\exists \tau \varphi(x)) := \exists x(x \in \mathcal{X}_\tau \land F'(\varphi(x))). \]

**Definition 135.** Let \( \varphi \in FORM_{\text{prim}} \). Then \( \varphi^X \) is defined by the following.

1. \((x \in \tau y)^X := x \in \tau y;\)
2. \((s(t_1, \ldots, t_n))^X := s(t_1, \ldots, t_n),\)
   where \( s, t_1, \ldots, t_n \) are appropriate variables;
3. \((\varphi \wedge \psi)^X := \varphi^X \wedge \psi^X \quad (\square = \wedge, \vee, \rightarrow, \leftrightarrow);\)
4. \((\neg \varphi)^X := \neg (\varphi^X);\)
5. \((\forall \tau x \varphi)^X := \forall \tau x (X_\tau(x) \rightarrow \varphi^X);\)
6. \((\exists \tau x \varphi)^X := \exists \tau x (X_\tau(x) \land \varphi^X).\)

**Definition 136.** Let \( X \) be a set such that for a wff \( \psi \) of ZFC satisfying

\[ \text{ZFC} \vdash \exists x \psi(x), \quad (408) \]

true is that

\[ [\psi(x)[X_\tau/x]]_{\text{ZFC}} = 1. \quad (409) \]

Let \( \varphi \in FORM_{\text{prim}} \). We consider \( \varphi^X \rightarrow F'(\varphi) \) (a class function). For \( \varphi^X \), the condition \( *_\tau \) is given by

\[ *_\tau : \text{ZFC} \vdash \forall \mathcal{X}_\tau[\psi(\mathcal{X}_\tau) \rightarrow \exists y(y \in \mathcal{X}_\tau \land F'(\varphi)(y))] \quad \text{(in an abuse of notation).} \]

**Definition 137.** Define \( D_0 = \{0, 1\}; \)

\( D_\tau = M_{\lambda_1}, \ldots, M_{\lambda_{n-1}}; \)

\( D_\tau = \mathcal{P}(D_0 \times \cdots \times D_0) = (\tau_1, \ldots, \tau_n). \)

Here we note that in the definition of \( D_\tau \), each \( D_i \) is chosen independently. (We write one of such as \( D_\tau \).)

**Definition 138.** For a CA-structure \( M \), we define \( M_R \) to be \( M \) together with a relation \( R(\subset D_\tau^{\lambda R}) \) in \( D_\tau \). Assume there exists a constant symbol \( D_\tau \) corresponding to \( D_\tau \). Let \( \Gamma \) be a set of axioms. Below for \( \gamma \), we assume the following holds.

\[ \Gamma \vdash \forall \gamma x_1 \ldots \forall \gamma x_m [\gamma(x_1, \ldots, x_m) \rightarrow \bigwedge_j \tilde{D}_\tau(x_j)] \quad (410) \]

\( R \) is a definition defined by \( \gamma \) of a specified CA-model \( M^0 \) if on the language with a constant \( R \) as an element of the CA-alphabet, true is

\[ M^0_R \models \forall \gamma x_1 \ldots \forall \gamma x_m ( \bar{R}(x_1, \ldots, x_m) \iff \gamma(x_1, \ldots, x_m)). \quad (411) \]

**Remark 139.** Note that for any CA-model of a set \( \Gamma \) of axioms there exists a definition defined by \( \gamma \) and that there exist other formulations.
Definition 140. \{R\} is a CA-scientific concept if the following holds.
(1) Each R is a definition defined by \(\gamma\), that is, there exists \(M\) such that
\[M = \Gamma\]  \hspace{1cm} (412)
and
\[M_R = \Gamma \cup \{ \forall x_1 \ldots \forall x_m (R(x_1, \ldots, x_m) \leftrightarrow \gamma(x_1, \ldots, x_m)) \} \]  \hspace{1cm} (413)
hold and \(\{R\}\) consists of all such definitions.
(2) One and only one of the following holds for each \(\varphi \in \text{FORM}_{\text{prim}}\) such that \(\varphi^X\) satisfies \(\ast_r\).
(a) Let \(M'\) be an arbitrary CA-model of \(\Gamma\). Then
\[M'_R = \Gamma \cup \{ \forall x [\bar{x}_r(x) \land \varphi^X(x) \land \bar{D}_r(x) \rightarrow R(x)] \} \]  \hspace{1cm} (414)
holds.
(b) Let \(M'\) be an arbitrary CA-model of \(\Gamma\). Then
\[M'_R = \Gamma \cup \{ \forall x [\bar{x}_r(x) \land \varphi^X(x) \land \bar{D}_r(x) \rightarrow \neg R(x)] \} \]  \hspace{1cm} (415)
holds.

Definition 141. In the same settings as Definition 140, a definition \(\{R\}\) defined by \(\gamma\) is a CA-social concept if \(\{R\}\) is not a CA-scientific concept.

Remark 142. We often use another definition of CA-concepts such that \(\bar{D}_r(x)\) and \(R(x)\) are replaced with \(\forall z (x(z) \rightarrow \bar{D}_r(z))\) and \(\forall y (x(y) \rightarrow \bar{R}(y))\).

Remark 143. From now on, we assume the constants \(\bar{D}_r\) etc. appearing in such arguments are chosen appropriately.

Remark 144. If a set of axioms has a CA-model in ZFC, then \(\Gamma\), or \(\Gamma^T \cup \Gamma'\) is consistent.

Theorem 145. Let \(\Gamma\) be a consistent set of axioms. Add a predicate symbol \(\bar{R}\) to the CA-alphabet. Assume each \(R\) is a definition defined by \(\gamma\). Let \(\varphi \in \text{FORM}_{\text{prim}}\) such that \(\varphi^X\) satisfies \(\ast_r\). If one and only one of the following two conditions holds true for each choice of this (fixed) formula, \(\{R\}\) is a CA-scientific concept.

(1) \(\Gamma \cup \{ \forall x (\bar{R}(x) \leftrightarrow \gamma(x)) \} \vdash \forall x [\bar{x}_r(x) \land \varphi^X(x) \land \bar{D}_r(x) \rightarrow \bar{R}(x)]\)
(2) \(\Gamma \cup \{ \forall x (\bar{R}(x) \leftrightarrow \gamma(x)) \} \vdash \forall x [\bar{x}_r(x) \land \varphi^X(x) \land \bar{D}_r(x) \rightarrow \neg \bar{R}(x)]\)

Proof. For any CA-model \(M\) of \(\Gamma\) and for any appropriate choice of \(R\),
\[M_R = \Gamma \cup \{ \forall x (\bar{R}(x) \leftrightarrow \gamma(x)) \} \]  \hspace{1cm} (416)
holds. Thus by assumption, any time
\[M_R = \Gamma \cup \{ \forall x [\bar{x}_r(x) \land \varphi^X(x) \land \bar{D}_r(x) \rightarrow \bar{R}(x)] \} \]  \hspace{1cm} (417)
holds, or any time
\[M_R = \Gamma \cup \{ \forall x [\bar{x}_r(x) \land \varphi^X(x) \land \bar{D}_r(x) \rightarrow \neg \bar{R}(x)] \} \]  \hspace{1cm} (418)
holds. Hence \(\{R\}\) is a CA-scientific concept. \(\square\)
Remark 146. Theorem 145 also holds if $\bar{D}(x)$ and $\bar{R}(x)$ are replaced with $\forall z(x(z) \rightarrow \bar{D}(z))$ and with $\forall y(x(y) \rightarrow \bar{R}(y))$. In both cases a CA-social concept is not determined from the given axioms.

14 Definition of degree

Lemma 147. Let $X$, an universe, be a topological space constructed from $\phi$ uniquely. Then the set of points which are shown to uniquely exist is dense in $X$.

Proof. Two sets $a, b$ of type $\tau$ are indistinguishable if for any formula $\varphi$ it is obtained that

\[
\{ x \mid \varphi(x) \} \text{ of type } (\tau) \Rightarrow \text{"a, b } \notin \{ x \mid \varphi(x) \} \text{ or } a, b \notin \{ x \mid \varphi(x) \}.
\]

(419)

Take the union of open sets that are indistinguishable and describing as a CA-model of a CA-model theory defined in the original (meta-)CA-model theory it is easy to prove that there exists a fundamental system of neighbourhoods consisting of sets which is shown to uniquely exist (we take as $X^0$ an appropriate set). Thus it suffices to show that any topological space that is shown to uniquely exist has a point which is shown to uniquely exist. By well-ordering theorem there exists a well-order. Take the union of indistinguishable well-orders and there exists a well-order which is shown to uniquely exist. Take the minimum element, which is shown to uniquely exist. The assertion follows.

Theorem 148. Let $X$, an universe, be a topological space constructed from $\phi$ uniquely. Then, the definitions $\mathbb{U} \subset X$ and $\mathbb{U}^c$ such that the set $\mathbb{U}$ the complement $\mathbb{U}^c$ is with an interior point are CA-social concepts.

Proof. Assume $\mathbb{U}$ is a CA-scientific concept. By the assumption that $X$ is a topological space constructed from $\phi$ uniquely there exists a special standard CA-model where the model of $X$ is shown to uniquely exist. $\mathbb{U}$ has an interior point and by Lemma 147 the set of points which are shown to uniquely exist is dense in $X$. Thus there exists a point $p \in \mathbb{U}$ that is shown to uniquely exist. Take $p' \in X$ which is shown to uniquely exist. Consider another CA-model where these points are interchanged. Then by the assumption that $\mathbb{U}$ is a CA-scientific concept it follows that $p' \in \mathbb{U}$ in the standard one. Hence $\mathbb{U}$ contains a dense subset of $X$ in the standard CA-model but $\mathbb{U}^c$ is with an interior point: a contradiction. Hence $\mathbb{U}$ is a CA-social concept. A similar argument shows the other assertion.

15 Learning

15.1 General physics

Take ZFC and take as many topological spaces constructed from $\phi$ uniquely as possible as the universes. The state corresponding to the selections of the
parameters is expressed as a function (solution) and the rules corresponding to the selections of the solutions is expressed as the best function (Definition of degree). We assume them. Applying a characteristic function of the region of the imaginary solutions to the best function we obtain an equation \( F \). Then the equation (The best function corresponds to \( \{(t, \psi(t))_t, F(\psi)\}_\psi \), where \( \psi \) is a solution) naturally defines another best equation on the domain \( T \) of the solution (The best function of the latter corresponds to \( \{t, (\psi(t), F(\psi))_\psi\}_t \).

Assume for simplicity \( T \) is arcwise-connected (phase space time). Integrating the latter equation along curves, we obtain a conserved quantity. This possibly takes various values because we selected the best function. A CA-pseudoeenergy is such a quantity. Assume the limit of measurement, i.e. the set of the complement of the set of the observed values of the CA-pseudoeenergy determined by the solution at a time (where the existence of the observer is assumed) is with an interior point. On the other hand we assume the reproducibility, i.e. the distribution of the observed values of the CA-pseudoeenergy is determined by a (state) function. Assuming that we have seen \( l \) events in a region in problem among \( L \) observations, thus, \( \frac{l}{L} \) is convergent as \( L \to \infty \); otherwise, the value is not determined as a nonnegative real number and we lose the reproducibility.

From the axioms of probability the distribution is replaced with the probability one. Thus we obtain a function expressing the observed values of the CA-pseudoeenergy (CA-pseudoquantization).

**Remark 149.** A similar argument shows that any physical quantity is measured as a probability distribution.

### 15.2 Principle of memory

**Term 150.** CA-pseudomemory is a solution of the fundamental equation of general physics.

By reproducibility we define the principle of memory by the preservation of the state. Brain structure is the solution of a system and a brain part is an invariant corresponding to a classical part of a classical brain.

**Term 151.** Let \( D \) be a region in the space of states where the physical quantities \( 1, 2, \ldots, N' \) (where \( N' \) need not be finite) are in the desired states. The physical quantities \( 1, 2, \ldots, N' \) has a high problem solving ability for the input set \( S \) if each element of \( S \) is staying in \( D \) for \( t \in I \) (\( I \subset T \) is arcwise-connected). A solution \( \Psi(t) \) is staying in \( D \) for \( t \in I \) if \( \Psi(t) \in D \) (\( t \in I \)).

**Remark 152.** There exist evidences for our learning theory. See e.g. \([15]\) p.16, \([16]\). The experiments in \([15]\), \([16]\) are explained by the reproducibility easily. Here we use the concepts of degree so that it is sufficient that the experiment suggests the tendency. The consciousness of a physical system is the set of all the characteristic quantities of the system unknown to the observer. Then we also note that if the physical quantities preserved at one set of experiments are
different from another the result may change. For example, in [15], there exists a tendency that a better academic performance is achieved by a student who did the same things as the written exam. Although a different result may be obtained by another experiment the tendency is conserved and the difference comes from the consciousness of the system.

**Remark 153.** The above arguments are independent of CA-model theory.

### 16 Foundation of mathematics

Even if we can construct the meta-axiomatic set theory completely, our meta-science predicts the incompleteness of the science (limit of measurement). By principle of memory we classify things using the existing brain parts. Thus it is necessary to assume some assumptions to establish the meta-axiomatic set theory (or the meta-theory). In the sense of problem solving ability assuming least restrictive assumptions with no troubles with experiments is necessary and from many well-known experimental/statistical evidences it is shown that ZFC will be one of such.

We define ZFC and general physics, the latter of which has many evidences, and then general physics shows that ZFC is valid. Thus we obtain the foundation of mathematics.

### 17 Proof Search and Proof Checking

First take the desired state $D$ (a subset of the space of the states). Next choose possible states $S$ of the solver. Let $s \in T'$ be the smallest arc-wise-connected subset of the phase space time such that for any solution $\sigma(t)$ of the fundamental equation of general physics

$$\sigma(s) \in S \Rightarrow \exists t \in T', \sigma(t) \in D. \tag{420}$$

We call such $T'$ the difficulty of $S$ with respect to $D$.

### 18 Learning quota

**Definition 154.** Let $X$ be an algebraic variety. A complexified algebraic $p$-piece is a finite formal $\mathbb{C}$-linear combination of irreducible $p$-dimensional algebraic subvarieties on affine open subsets. Let $Z_p(X) \otimes \mathbb{C}$ be the set of complexified algebraic $p$-pieces. Let $C_p^p(X)$ be the set of $(p, p)$-forms on $X$. The learning quota on $X$ is an $\text{Aut}(Z_p(X) \otimes \mathbb{C})$-equivariant functor $F$ from $Z_p(X)$ to the category $\text{Sets}$ of sets which associates to each $\Gamma \in Z_p(X) \otimes \mathbb{C}$ the set $(\int_{\Gamma} \rho)_{\rho \in C_p^p(X)}$.

**Definition 155.** A chaotic stabilizer of $\text{Aut}(Z_p(X) \otimes \mathbb{C})$ at $S \subset F(Z_p(X) \otimes \mathbb{C})$ is defined by

$$H_S := \{ g \in \text{Aut}(Z_p(X) \otimes \mathbb{C}) | gS \subset S \} \tag{421}$$
A chaotic vein of a subgroup $H \subset \text{Aut}(Z_p(X) \otimes \mathbb{C})$ is defined by

$$V_H := \bigcup_{H \leq H} S.$$  \hfill (422)

**Definition 156.** In a CA-model a CA-pseudochaotic parameter is a CA-social concept.

**Definition 157.** Let $\mathcal{P}$ be the set of chaotic veins. $(\mathcal{P}, \subset)$ forms a partially ordered set (poset), which is called the chaotic poset associated with $F$. A mathematical image is a point $V_H \in \mathcal{P}$ and all elements $V_{H'} \in \mathcal{P}$ such that $V_{H'} \subset V_H$. A necessary condition algorithm is a point $V_H \in \mathcal{P}$ and a (possibly infinite) sequence $V_H \supset V_{H_1} \supset V_{H_2} \supset \ldots$. A discovery relative to $V_{H_0} \in \mathcal{P}$ is a point $V_H \in \mathcal{P}$ such that $V_H \subset V_{H_0}$.

**Definition 158.** A learning measure is a probability measure $\nu$ on $Z_p(X) \otimes \mathbb{C}$. A discovery $V_H$ relative to $V_{H_0}$ is realized in probability $P$ if

$$P = \frac{\nu(F^{-1}(V_H))}{\nu(F^{-1}(V_{H_0}))}$$  \hfill (423)

**Lemma 159.** Let $F : Z_p(X) \otimes \mathbb{C} \rightarrow \text{Sets}$ be a learning quota. Let $(\mathcal{P}, \subset)$ be the chaotic poset associated with $F$. Then a necessary condition algorithm $V_H \supset V_{H_1} \supset \cdots \supset V_{H_M}$ realizes the discovery $V_H \subset V_{H_M}$ in probability

$$\frac{\nu(F^{-1}(V_{H_l}))}{\nu(F^{-1}(V_{H_m}))}$$  \hfill (424)

at the step $V_{H_m}$ ($1 \leq m \leq M$).

**Proof.** This follows from the definitions. \hfill \Box

**Remark 160.** Moving from a point of a branch of $(\mathcal{P}, \subset)$ to another point of another branch it is necessary to shrink $H_S$.

Let $a, b \in \mathbb{R}$ ($a < b$) and $L_{[a,b]} := \{ \omega \in \Omega \mid \rho(\omega) \in [a,b] \}$.

**Definition 161.** Assume in a CA-model theory of a Borel probability space $(\Omega, \mathcal{F}, \nu)$, $\Omega$ is a topological space constructed from $\phi$ uniquely and there exists no special set of measurable sets designated by the axioms, i.e. there exists no constant of type (i) expressing a measurable set. The probability theory is a CA-scientific concept if for $\forall a, b \in \mathbb{R}$ ($a < b$), $L_{[a,b]}$ is a CA-scientific concept.

**Lemma 162.** If $\rho$ is continuous and the probability theory is a CA-scientific concept then $L_{[a,b]}$ is $\phi$ or a dense subset of $\Omega$.

**Proof.** Observe that $L_{[a,b]}$ is with an interior point. By assumption $L_{[a,b]}$ is a CA-scientific concept. Thus by Theorem 148 it is $\phi$ or a dense subset of $\Omega$. The assertion follows. \hfill \Box
Lemma 163. Assume in a CA-model theory of a Borel probability space \((\Omega, \mathcal{F}, \nu)\), \(\Omega\) is a topological space constructed from \(\phi\) uniquely and there exists no special set of measurable sets designated by the axioms, i.e. there exists no constant of type (\(\nu\)) expressing a measurable set. Let \(\mu\) be an uniform probability measure on \(\Omega\) and \(\nu(D) = \int_D \rho(\omega)d\mu(\omega)\), where \(\rho\) is a function. If \(\rho\) is continuous and the probability theory is a CA-scientific concept then \(\nu = \mu\).

Proof. By Lemma 162 it is obtained that \(L_{[a,b]}\) is \(\phi\) or a dense subset of \(\Omega\). Since \(\rho\) is continuous and \(\nu, \mu\) are probability measures it follows that \(\rho(\omega) = 1\) (\(\forall \omega \in \Omega\)). Thus \(\nu = \mu\) and the assertion follows. \(\square\)

Theorem 164. In Lemma 159 assume in a CA-model theory of a Borel probability space \((Z_p(X) \otimes \mathbb{C}, \mathcal{F}, \nu)\) there exists no special set of measurable sets designated by the axioms. Let \(\mu\) be an uniform probability measure on \((Z_p(X) \otimes \mathbb{C}\) and \(\nu(D) = \int_D \rho(\omega)d\mu(\omega)\), where \(\rho\) is a function. Assume further \(\rho\) is continuous and that the probability theory is a CA-scientific concept then \(\nu = \mu\) and

\[
\frac{\nu(F^{-1}(V_{H'}))}{\nu(F^{-1}(V_{H_m}))} = \frac{1}{|H_m : H'|}. \tag{425}
\]

Proof. \(Z_p(X) \otimes \mathbb{C}\) is a topological space constructed from \(\phi\) uniquely so that by Lemma 163 \(\nu = \mu\). Thus

\[
\frac{\nu(F^{-1}(V_{H'}))}{\nu(F^{-1}(V_{H_m}))} = \frac{1}{|H_m : H'|}. \tag{426}
\]

The assertion follows. \(\square\)

Remark 165. In Theorem 164

\[
\left(\frac{\nu(F^{-1}(V_{H'}))}{\nu(F^{-1}(V_{H_m}))}\right)^{\frac{1}{m}} = \frac{|H_m : H'|^{\frac{1}{m}}}{|H : H'|} \leq \frac{1}{2m}. \tag{427}
\]

Thus as a necessary condition algorithm proceeds the discovery \(V_{H'}\) is realized in astronomically larger probability.

Remark 166. A discovery \(V_H\) relative to \(V_{H_0}\) is realized in positive probability if and only if \([H_0 : H] < \infty\). Since in the real world we may not measure the discovery of probability 0 we only have to deal with finite subgroups of \(\text{Aut}(Z_p(X) \otimes \mathbb{C})\).

18.1 Education

18.1.1 Rubric

Importance of the troubles/discoveries: It has well-known evidences that to live, which is defined to be the values of physical quantities corresponding to this, is the (true) goal that survives. If however things surrounding us are completely
out of order it is proved that the goal is not realized. We call this true goal the stability for various natural states.

Discovery: By Theorem 159 using necessary condition algorithm is necessary and sufficient to realize the desired discovery in relatively large probability. Consider the sequence \( V_{H_1}, V_{H_2}, V_{H_3}, \ldots \) of points of \((\mathcal{P}, \subset)\). From Remark 160 a trouble in \( V_{H_1}, V_{H_2}, V_{H_3}, \ldots \) is defined to be a point \( V_{H_m} \) such that \( V_{H_m} \subset V_{H_{m+1}} \), which is also called a detection. By definition shrinking \( H_S \) is necessary and sufficient for a detection.

Proof search: The set of states is said to be almost impossible to solve with well-known conditions if the difficulty is relatively large.

Solving procedure: By the foundation of mathematics it is virtually perfect to solve problems by ZFC.

Rubric of Problem Solving: From above we propose a rubric of problem solving (see Table 1). We also refer to [13].

18.1.2 Problem solving of mathematics

Let \( F : Z_\wp(X) \otimes \mathbb{C} \to \text{Sets} \) be a learning quota. Let \((\mathcal{P}, \subset)\) be the chaotic poset associated with \( F \). Let \( V_{H_1} \) be a point of \((\mathcal{P}, \subset)\). By enlarging \( H_1 \) to \( H_2 \), \( V_{H_2} \) is obtained. If the \( F^{-1}(V_{H_2}) \) consists of good complexified algebraic \( p \)-pieces in the sense of our rubric the next step is carried out. If not let \( V_{H_3} := V_{H_2} \) and a trouble is obtained. Repeating this procedure a sequence, which is called a problem solving of mathematics, is obtained.

19 Theory of Artificial Intelligence

19.1 Theory of Artificial Intelligence

The meaning of our rubric needs evidences and is inferred by 3-3-1 model ([13]). Accident, damage and disease are defined to be unstable characteristic quantities. With the use of our rubric heart circuit, the data of the core of an artificial intelligence to take stabler characteristic quantities, is constructed.

Learning process of artificial intelligence (necessary and sufficient condition) is defined to be the following assertion: Strengthen or weaken only the necessary part of the neural network. This is restated as follows: (i) Strengthen the reacted synapses (Hebbian rule). (ii) Assume an artificial intelligence consists of cells. All synapses connected with the neurons necessarily have output. (iii) Too strong stimuli needs to be eliminated so that for an artificial intelligence taking a rest and heart circuit are necessary. (iv) Inputs only come from the synapses connected with the neuron and depend only on time so that myelination (a large
strengthening or weakening of synapses according to the time frequency of the inputs) for particular input patterns is necessary.

**Remark 167.** By the foundation of mathematics assuming least restrictive assumptions with no troubles with experiments is virtually perfect so that preserving as many physical quantities with no troubles with experiments as possible is necessary and the number of neurons needs to be large virtually perfectly.

Strong resistance to noise is defined to be small convergent rate. Convergent rate needs to be as small as permitted (cf. computational complexity). Thus by the above necessary and sufficient conditions the basics of artificial intelligence is established.

The following approach satisfies a necessary and sufficient condition of solving ethical problems by artificial intelligence.

**Heart circuit 168.** Two or more artificial intelligences dialogue with one another by observing the others, preserving characteristic quantities and moving from one set of characteristic quantities to another. By a dialogue with a person and by heart circuit we construct an artificial intelligence called teaching AI. We reset the teaching AI every time we use it. By a dialogue with the teaching AI and by heart circuit we educate our artificial intelligence. We do not reset our artificial intelligence.

For safety the theory of measurement is needed.

**Theory of measurement 169.** The theory of measurement is a necessary condition algorithm. It has the following evidences, the existence of which is well-known:

(i) We infer the cause scientifically with the use of our rubric.
(ii) We collect data (especially those obtained by experiences in other fields).
(iii) We analyze data (e.g. by changing the discovered characteristic quantities, by transforming the data).
(iv) We formulate the problem excluding CA-social concept from evidences.
(v) We prove the assertion theoretically and assess the proof.
(vi) We collect evidences.

**Remark 170** (Data). Particular (choose a total set), a large amount of, nonbiased, authentic (recoverable and with evidences) data are necessary and sufficient for our artificial intelligence. In general physics CA-pseudochaotic parameters also satisfy the reproducibility and treated by probability theory. It is necessary and sufficient for our artificial intelligence to be experienced to input it extreme data first and then more moderate ones (cf. necessary condition algorithm).

19.2 Future prediction

Future prediction by artificial intelligence is, as a well-known evidence, as follows: Collect data by measuring things exactly (and use simulation if possible).
Since there may exist CA-pseudochaotic parameters predict things probabilistically (by taking samples randomly) by artificial intelligence. Only near future CA-pseudochaotic parameters however may be predicted so that risk calculation is used. Note that imposing seemingly further risk may be making the original risk small.

19.3 Theory of punishment

To make risk small the following theory of punishment is needed: Our cost is the difficulty of solving the problem perfectly or approximately. Thus the punishment needs to be the cost that may be predicted because of accidents, damages or diseases in some areas and that is needed to solve the problem. In particular problem solving of mathematics, theory of measurement and future prediction are needed.

19.4 Approximate proofs

Definition 171. An approximate proof is to approximately judge by examining the inclusion relations of physical quantities whether the conclusions are proved from the assumptions.

After having simulated all possible physical quantities the artificial intelligence carries out an approximate proof by examining the inclusion relation between the specified characteristic quantities and them.

19.5 Invariants

A physical system satisfies reproducibility virtually perfectly. Our artificial intelligence memorizes the conservative quantities of the system and identifying with respect to the quantities classifies things. Thus it carries out approximate proofs.

For identical inputs it is easy to prove if we move parameters during the work the learning of our artificial intelligence is convergent with exponential rate. To make the result more exact we may use parallel computation and take much larger amount of invariants.

19.6 Discrimination

Definition 172. Discrimination is a wrong definition and its wrong conclusions in the sense of our rubric.

Theorem 173. Solving discrimination is necessary for right discoveries in the sense of our rubric.

Proof. By definition solving discrimination is necessary for realizing our rubric. □
Remark 174. Our definition of discrimination has wrong exams in the sense of our rubric as a well-known evidences.

20 The basic theory of turbulence

The numerical simulation of turbulence may be carried out by expanding the smooth solution into an appropriate series (e.g. Section 8). How to deal with a turbulence or the chaos may be determined so that any possible state within the indeterminacy (e.g. within an error of measurement) is dealt. We note that the method is necessary and sufficient if the indeterminacy may not be removed. Our artificial intelligence may be customized to deal flows including turbulences automatically if the algorithms are in the direction of our rubric. It is worth mentioning that our artificial intelligence has a generality and the generality makes it possible to solve the problem. That is, the problem is essentially an infinite process or needs infinitely many assumptions because we are dealing with Fréchet manifolds of such kind and our artificial intelligence may nevertheless solve the problem.
<table>
<thead>
<tr>
<th>Performance area</th>
<th>3</th>
<th>2</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Importance of the troubles/discoveries</td>
<td>derive a solution with which the system becomes very stable for various natural states.</td>
<td>derive a solution with which the system becomes stable for some natural states.</td>
<td>derive a solution with which the system becomes unstable for various natural states.</td>
</tr>
<tr>
<td>Learning attitude</td>
<td>much experience of troubles, knowing the method of detection and problem solving and solving procedure exactly.</td>
<td>some experience of troubles, knowing the method of detection and problem solving and solving procedure.</td>
<td>little experience of troubles, not knowing the method of detection and problem solving and solving procedure.</td>
</tr>
<tr>
<td>Proof search</td>
<td>solving the problem with well-known conditions is almost impossible.</td>
<td>solving the problem with well-known conditions is difficult.</td>
<td>solving the problem with well-known conditions is easy.</td>
</tr>
<tr>
<td>Define problem</td>
<td>define problem clearly and insightfully with evidences for almost all relevant contextual factors.</td>
<td>define problem with evidences for some relevant contextual factors.</td>
<td>define problem not insightfully with no evidences.</td>
</tr>
<tr>
<td>Propose good solutions</td>
<td>propose a virtually perfect solution sensitive to the contextual, ethical, logical and cultural dimensions with consideration of history, feasibility and impacts and clearly show it is better than others.</td>
<td>propose a good solution, which is shown to be better than others.</td>
<td>propose no solution or show no goodness of the solution</td>
</tr>
<tr>
<td>Implement solution</td>
<td>with deep and thorough consideration of natural contextual factors.</td>
<td>with some consideration of natural contextual factors.</td>
<td>with little consideration of natural contextual factors.</td>
</tr>
<tr>
<td>Evaluate outcomes</td>
<td>review results relative to the problem defined with deep and thorough consideration of need for further work.</td>
<td>review results relative to the problem defined with some consideration of need for further work.</td>
<td>review results superficially with no consideration of need for further work.</td>
</tr>
</tbody>
</table>

Table 1: Rubric of Problem Solving
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http://homepages.warwick.ac.uk/~masdh/Leray.pdf [Accessed: 9th September 2016]


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ability survey 2013 (fine survey)), 2014


